HOW TO DETERMINE SOLAR INSOLATION (W-Hr/m^2)

USING GRAPHICAL RECTANGULAR, TRAPAZOIDAL, PARABOLIC

AND INTEGRATION METHODS

A REFRESHER

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ABSTRACT

At times there may be the need to determine the total quantity of solar energy (watts per square meter per day, W-day/m^2), that is available during a day's worth of sunshine (solar insolation), based on instantaneous solar irradiance (watts per square meter, W/m^2) measurements.

Four 'area under the curve' methods are discussed for determining insolation from measured irradiance data:

- Graphical sum of rectangles;
- Graphical sum of trapezoidals;
- Graphical sum of parabolic bounded areas (Simpson's Rule); and
- Integration (Calculus) of trendline equations.

Each method can provide progressively more accurate results.

This paper summarizes the expectation that the Integration (Calculus) of trendline equation is the most accurate (provided the trendline correlation factor is close to unity (1)) and the quickest of the methods to use. The Integration formula is presented in a user friendly 'cook book' procedure.

A detail discussion of each method is presented in Appendix A, complemented with example calculations of how to determine the four methods.

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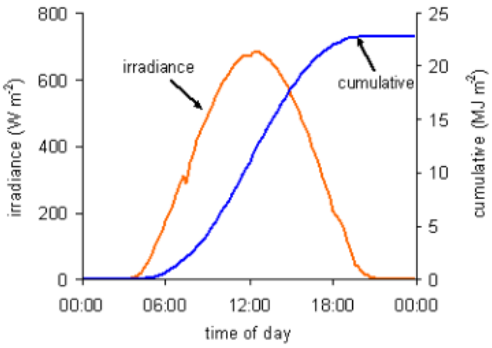
INTRODUCTION

At times there may be the need to determine the total quantity of solar energy (watts per square meter per day, W-day/m^2), that is available during a day's worth of sunshine (solar insolation), based on instantaneous solar irradiance (watts per square meter, W/m^2) measurements recorded over time.

Solar irradiance measurements (except for extremes at the north and south pole latitudes), generally starting right before sunrise and ending right after sunset, is essentially zero (the earth's shadow blocks the sun's rays), and peaks around noontime. Terrestrial solar irradiance, before striking the earth's surface, passes through varying thickness of atmosphere (the air mass or AM which equals 1 at noon) depending on the time day. Wherein low angle sunrise and sunset sunshine, passes through many air mass thicknesses, part of the irradiance is absorbed by the atmosphere (by water, CO_2, ozone absorption and scattering, for example), with the most intense terrestrial solar irradiance at noon time, when the sun is overhead and has the thinnest air mass layer to travel through. In addition, irradiance intensity also varies depending on how far the earth is from the sun (aphelion – furthest from the sun, and perihelion, closest).



Instantaneous terrestrial solar irradiance measurements (recorded at an earth's surface test location) when plotted on a graph against the time interval of measurement, generally will show a characteristic parabolic shaped curve (see figure to the right). The area under that curve is equivalent to total energy measured in watts per square meter per unit of time, W/m^2 x time (day), which equates to the total quantity of energy (insolation) accumulated over the day at the measurement location. The greater the number (shorter) time interval measurements taken over a test period will result in a more accurate curve. There are some measurement tools that take data measurements continuously. In addition some data logger measurement tools also provide for integrators that automatically assess the area under the curve.

There are different techniques in the absence of technology, for determining the insolation or area under the curve. Three manual calculated techniques include graphical procedures (rectangle, trapezoidal, Simpson Rule) and a fourth, an analytical technique (integration of a curve model, a little bit of calculus – but not to worry, this paper simplifies the process of calculation if you haven't used calculus in a while). There are various software programmes that can also assist with the area determination. This paper illustrates how the techniques work and a comparison of their respective results, and a suggestion that in the absence of more rigorous tools, the analytical Calculus technique is a quick and convenient method to use.

Appendix A is a discussion of the derivation of each area under the curve determination technique, an in-depth understanding of which, is not essential for employing the area calculations described in this paper.

COLLECTING AND PLOTTING THE DATA

Assume a solar developer is in need of understanding the quantity of solar energy insolation (W-day/m^2) at a particular location, and on a particular day. While only an example, this discussion illustrates the energy determination technique principles that are applicable on a larger scale.

Assume the solar developer uses a pyranometer to measure solar irradiance data, and manually records the data every 15 minutes, from sunrise (7:00 a.m.) to sunset (7:00 p.m.). The data is then graphed (irradiance values on the y axis, and time values on the axis). Example recorded irradiance data is reported in Appendix B Data Table and in Appendix C, which are graphs of the data that illustrate the irradiance plotted against time (note the characteristic parabolic curve profile).

Using Microsoft™ Excel™ spreadsheet, the parabolic looking curve can be approximated by determining a trendline equation. Trendlines determined for Appendix C graphs are illustrated in Appendix D. The trendline equation (which varies depending on the time interval units recorded -- plotted on the x axis of the graphs) for the data point numbered graph is:

$$y = 0.0123x^3 - 2.7184x^2 + 106.87x - 172.91,$$

which is a 3rd order polynomial equation; and $R^2 = 0.9941$, where,

y = predicted irradiance (W/m^2);
x = recorded time data number (0, 1, 2, … 50);
R^2 = (correlation factor) measure of how close the trendline matches the actual data. The closer the factor is to 1, the closer the trendline approximates the actual measured data.

The trendline equation for the x axis time line recorded in time of day time is:

$$y = 10882x^3 - 34234x^2 + 26934x - 5282.3,$$

which is a 3rd order polynomial equation; and $R^2 = 0.9941$, where,

y = predicted irradiance (W/m2);
x =time of day when data was recorded (7:00 AM, 7:15 AM, …7:00 PM);
.
The above equations determine the same predicted irradiance (y) results, even though each equation respective terms vary, depending on the x axis time units chosen.

DETERMINIG THE AREA UNDER THE CURVES –

TOTAL ENERGY PER 12 HOUR MEASURMENT PERIOD (INSOLATION)

METHOD 1: SUM OF THE AREA RECTANGULES (generally least accurate)

The sum of the area rectangles method approximates the area under the irradiance vs time curve using the *inner* rectangles (each rectangle is inside the curve) bounded by the x (time) axis and the curve.

There are three rectangles that could be summed (outer, inner and mid-point rectangles) as illustrated in Appendix E (detail discussion in Appendix A).

Appendix E table illustrates the area under the curve for each rectangle location (inner, outer, mid-point). Each rectangle area is determined by multiplying its base times it height, and then summing all the rectangles. The base is a constant 15 minute interval (or $\Delta t = 0.25$ hours). The height is the irradiance measurement take off the curve associated with a given time interval and varies with which of the outer, inner or mid-point rectangles are used in the area determination. As shown the rectangle method resulted in an energy under the curve of 7730.75 W-Hr/$m^{2,}$, or 7.73 peak sun hours (where 1000 W-Hr/m^2 is equivalent to one peak sun hour; 7730.75/1000=7.73). In this example all three rectangle methods give the same answer because of the 0 end point irradiance measured starting and ending conditions for all three rectangles.

METHOD 2: SUM OF THE AREA TRAPEZOIDALS (generally more accurate than rectangle method)

The sum of the trapezoidal method approximates the area under the irradiance vs time curve using sum of trapezoids bounded by the x (time) axis and the curve.

An illustration of the trapezoidal method is shown in Appendix F (detail discussion in Appendix A).

Appendix F table illustrates the area under the curve for each trapezoid. Each trapezoid area is determined by multiplying its base (Δt, 0.25 Hrs), times the sum of (2 times the first side of the first trapezoid – being the first irradiance measurement, plus the sum of all the other irradiance measurements up to the last measurement, plus 2 times the last side of the last trapezoid, being the last irradiance measurement). As shown the trapezoidal method resulted in an energy under the curve of 7730.75 W-Hr/m2, or 7.73 peak sun hours (where 1000 W-Hr/m2 is equivalent to one peak sun hour; 7730.75/1000=7.73). This particular example total energy coincidentally is the same as the rectangular method.

METHOD 3: SIMPSON'S RULE SUM OF THE AREA OF PARABOLAS (generally more accurate than the Trapezoidal rule)

In Simpson's Rule, parabolas are used to approximate each part of the curve. This is very efficient since it's generally more accurate than the other (rectangle or trapezoidal) numerical methods. Divide the area into n equal segments of width Δx (Δt, 0.25 hours). The approximate area is given by the following.

Simpson's Rule

$$\text{Area}= \int_a^b f(x)dx \approx (\Delta t/3)\ (y_0+4y_1+2y_2+4y_3+2y_4\ldots+4y_{n-1}+y_n)$$

where $\Delta t=(b-a)/n$ = 0.25 hours in the example

Note: In Simpson's Rule, n must be an EVEN number.

Appendix G table illustrates the area under the curve for Simpson's Rule. As shown Simpson's Rule method resulted in an energy under the curve of 7765.23 W-Hr/m2,, or 7.77 peak sun hours (where 1000 W-Hr/m2 is equivalent to one peak sun hour; 7765.23/1000=7.77).

METHOD 4: INTEGRATION METHOD (generally most accurate)

In regard to the Integration Method, the trendline curve formula is integrated as follows:

$f(t) = y = at^3 + bt^2 + ct + d$ (the trendline formula, a 3rd order polynomial in this example).

Integrate as follow:

$$F(t) = dy = \int(at^3 + bt^2 + ct + dt^0)dt = at^{(3+1)}/(3+1) + bt^{(2+1)}/(2+1) + ct^{(1+1)}/(1+1) + dt^{(0+1)}$$

$$= a/(4)t^4 + b/(3)t^3 + c/(2)t^2 + (d/1)t^1$$

In otherwords, for the polynomial trendline, for each term increase the exponential by 1 and divide the exponent result into the component of the equation.

For the example,

$y = 0.0123t^3 - 2.7184t^2 + 106.87t - 172.91(t^o = 1)$;

$F(t) = (0.0123/4)t^4 - (2.7184/3)t^3 + (106.87/2)t^2 - 172.91t^1$

And for $\int_{t=1}^{t=50} ((0.0123/4)t^4 - (2.7184/3)t^3 + (106.87/2)t^2 - 172.91t^1)$

$$= ((0.0123/4)(50)^4 - (2.7184/3)(50)^3 + (106.87/2)(50)^2 - 172.91(50)^1) -$$

$$((0.0123/4)(1)^4 - (2.7184/3)(1)^3 + (106.87/2)(1)^2 - 172.91(1)^1)$$

= 30894 – 120 = 30,774 W-min/m^2

Dividing by 4 (since the measurements were in 15 minute increments) results in…

= 7793 W-Hr/m^2

Comparted to the graphical methods, the integration method is much faster.

COMPARING THE METHODS

A comparison of the four methods shows:

Method	Total Energy, W-Hr/m^2
Rectangle	7731
Trapezoidal	7731
Simpson Rule	7765
Integral Calculus	7793

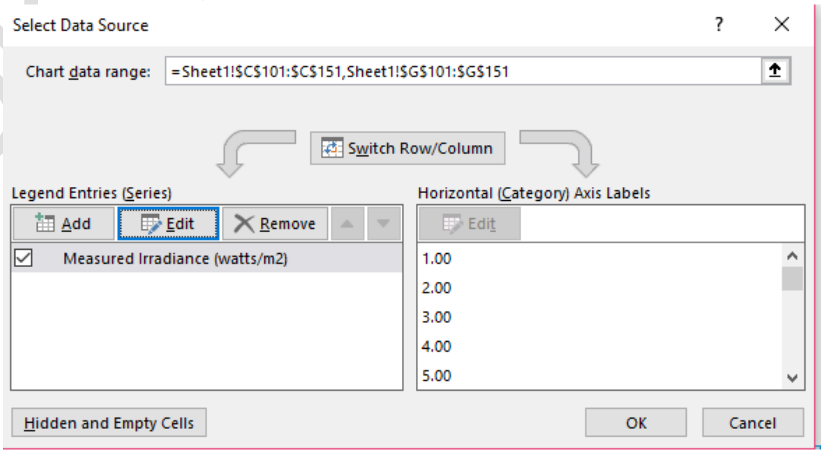
INTEGRATION STEP-BY-STEP METHOD

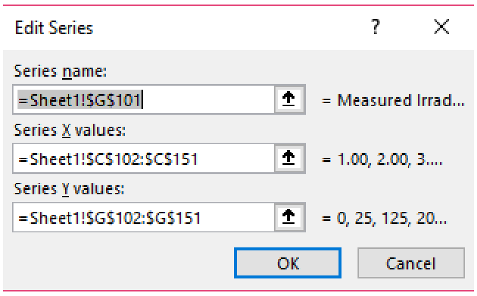
Step 1: Record instantaneous irradiance measurements (say from a pyranometer), and record the time interval between each measurement. (Select a convenient time interval of measurement, say every 15 minutes, or quarter of an hour). Measure from sunrise to sunset. Take note of units. In this example irradiance is measured in Watt/m^2 and time intervals in minutes, so the recorded data point is number 1 and ending at data point number 50 over a 12 hour period, from 7:00 a.m. (sunrise) to 7:00 p.m. (sunset). The more data points recorded the more accurate the plotted data, equally the more time consuming the data collection.

Step 2: Plot by hand or use graph software (such as Microsoft™ Excel™), irradiance on the y axis and the time on the x axis. Appendix B illustrates a table of recorded example irradiance data over time.

Appendix C illustrates a plot of the data using Microsoft™ Excel™ (plotting both irradiance and time units of 15 minute interval data points and time of day).

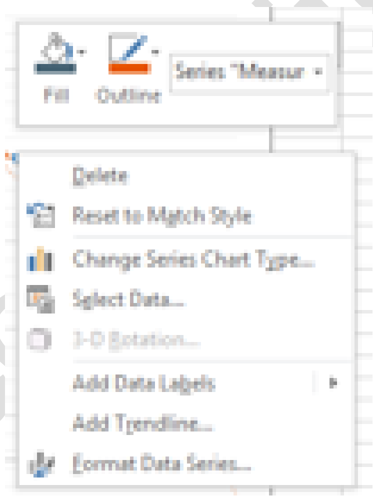
Graph data input is illustrated below:



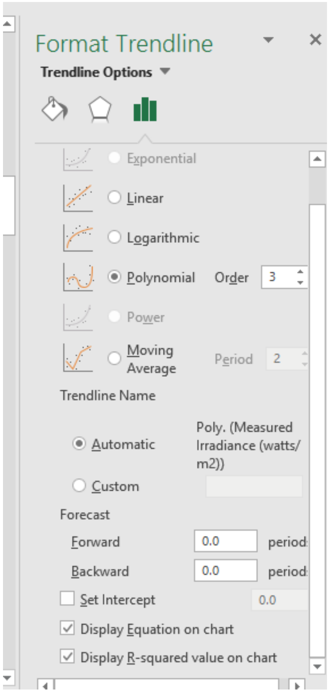


Step 3. Using Micosoft™ Excel™ graphed data, determine a trendline equation for the plotted data. Appendix E illustrates such trendline equation for the Appendix A example data. A trendline equation should be selected that provides a good correlation factor (R2 ≈1.0, the closer to 1, the better the correlation). Typically for a fairley uniform parabolic structured curve, a 3rd order polynomial equation will give a nice fit.

Click on the plotted data curve and right click to bring up 'Add Trendline".



Select trendline options, typically polynomial and 3rd order, display equation on chart and display R-squared value on chart.



Step 4: After accepting a trendline having a R^2 value close to one, integrate the equation as follows:

Unintegrated equation:

$y = at^3 + bt^2 + ct + dt^0$ (note the last component is dt^0, usually t^0 is not written since a number raised to the zero power is equal to one or $t^0 = 1$); where…

y = predicted irradiance in Watt/m^2

t = unit of time (can be each individual data point number, 1,2,3, etc to n, or time of day 7:00 a.m., 7:15 a.m, etc. The trendline equation terms will be different depending on what units of measurement are used to determine the equation.

a, b, c and d are constant coefficients that the trendline equation has determined. Caution: the above example indicates the equation has all positive (+) values combining the coefficients, however, the trendline may have a mixture of positive (+) and negative (-) values… so be careful to maintain the same signs. (For example, a trendline might be: $y = -at^3 – bt^2 + ct – dt^0$)

Integrate the above equation, by increasing the exponent of each t value by 1 and divide each component by that factor: For example…

$[a/(3+1)]t^{(3+1)} + [b/(2+1)]t^{(2+1)} + [c/(1+1)]t^{(1+1)} + [d/(0+1)]t^{(0+1)}$

$(a/4)t^4 + (b/3)t^3 + (c/2)t^2 + (d/1)t$

Substitute into the integrated equation the last data point (keeping units consistent) and in this case data point 50, and determine the integral.

Do the same with the first data point, 1.

Subtract from the (50) data point result, the (1) data point result, and the resultant number is the area under the curve (Watts-quarter hour)/m^2, because in this example, data was taken every 15 minutes, or quarter of an hour. The result is divided by 4, to convert to Watt-Hour/m^2.

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APPENDIX A

DERIVATION OF AREA UNDER A CURVE

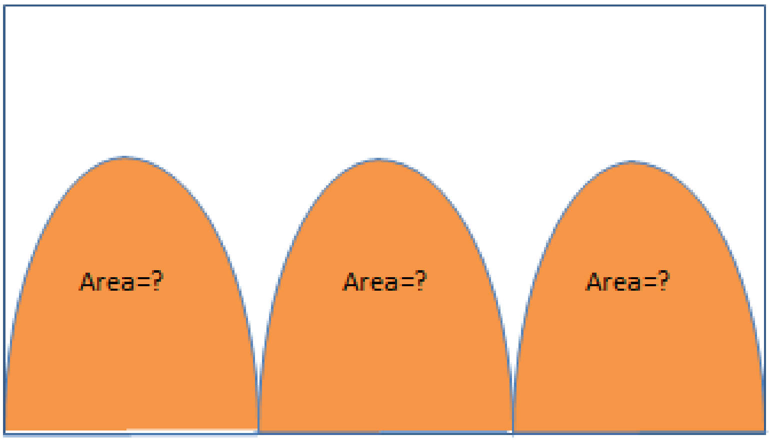
GRAPHICAL METHODS (RECTANGLE, TRAPEZOID, SIMPSON'S RULE)

AND

INTEGRATION (CALCULUS)

The Area under a Curve

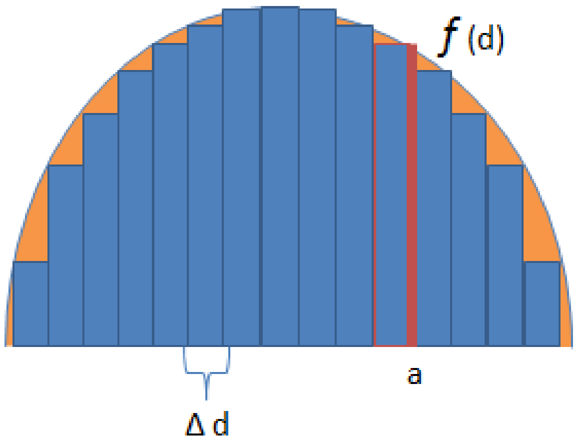
Assume a museum is to be built with three archways. These archways are to be covered in tiles. What is the area under each archway ('area under the curve') which will determine how many tiles are required? (*The arch problem*).



There are at least four ways to determine the area:

1. Using three different graphical approximation techniques:
   1.1. Finding areas of rectangles,
   1.2. trapezoids or
   1.3. parabolic boundaried areas – ('Simpson's Rule'), or
2. Using (Calculus) integration

Prior to the development of calculus **integration**, the engineer could only **approximate** the area under the curve, by dividing the area into many rectangles (the more rectangles, the more accurate the estimate) and adding the areas of all the rectangles (a tedious manual graphical determination):



To determine the area of a rectangle, multiply its width (Δd, delta d) times its height (a, which is determined from either direct measurement or from the equation, *f*(d), which is short hand notation for the *function* of the curve), as shown in the illustration.

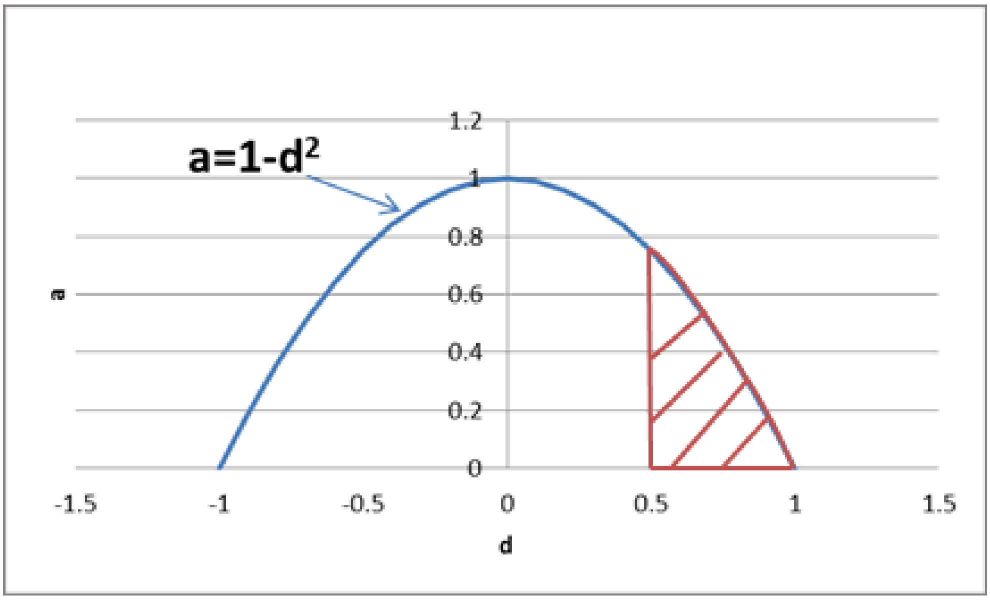
The above illustration is based on using 'inner rectangles', being rectangles located within the archway curve. Area of rectangle = height x base; *f*(d) * Δd or for the example rectangle, Area = a * Δd.

There are two other rectangle methods to determine area, 'outer rectangles' and 'mid-point rectangles', as illustrated below.

Graphical Method Approximation Based On Rectangles

1. Using 'outer rectangle' graphical method, determine the area under the curve $a = 1 - d^2$ between d = 0 and d = 1, for # = 5 (number of rectangles), based on sum of areas of rectangles method.

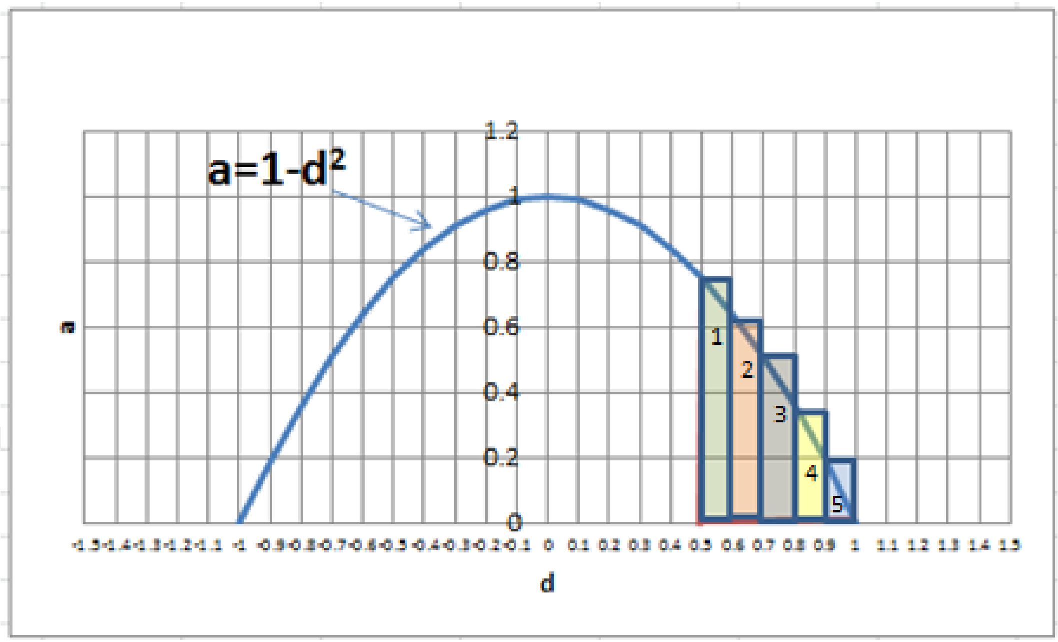
The approximate graphical area of interest is illustrated below:



Because # = 5 (the number of rectangles to be considered), the width of each ‘outer rectangle’ is:

d =Δd = 0.1   [(1-0.5)/5]

The sum of the areas of each 5 ‘outer rectangles’ are illustrated as follows:



The height (a) of each outer rectangle is determined by the curve equation or function (*f*(d)) value for the selected *d* value.

For d = 0.5, since $a = f(\text{d}) = 1-d^2$, outer rectangle number 1 has height (a) determined by:

$f(0.5) = 1 - (0.5)^2 = 0.75$

Its area is:

$\text{Area}_{(\text{outer rectangle no. 1})} = 0.75(\text{a}) * 0.1(\Delta\text{d}) = 0.075$ square unit area

For outer rectangle number 2, its height (a_2) is:

$f(0.6)_2 = 1 - (0.6)^2 = 0.64$

and so on, and for the fifth outer rectangle,

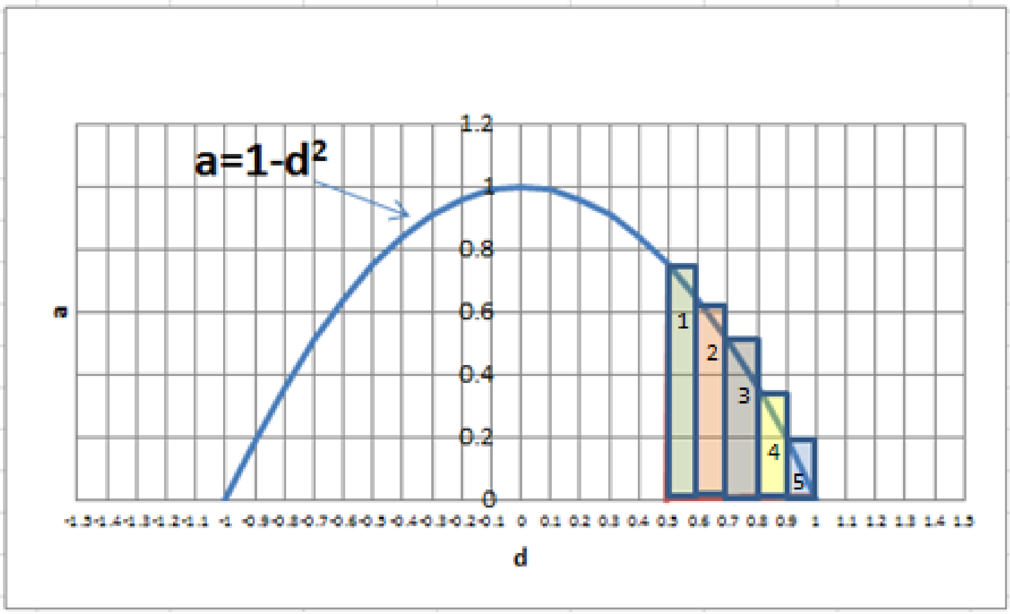
$f(0.9)_5 = 1 - (0.9)^2 = 0.19$

Summing up the five outer rectangle areas results in:

$$A = \sum_{\#=1}^{\#=5} A(\#) = (0.75)(0.1)+0.64(0.01)+0.519(0.1)+0.36(0.1)+0.19(0.1)$$

$$= 0.245 \text{ square units}$$

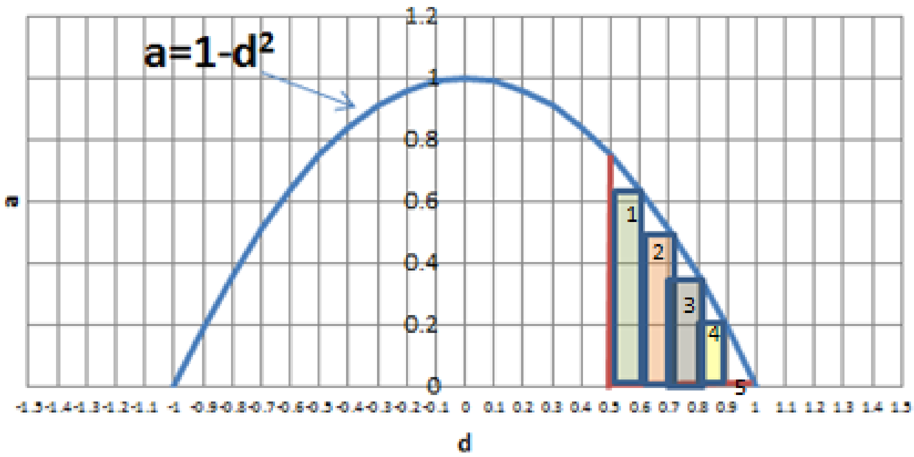
A more accurate approximation of the rectangle areas, would be based on determining the area of **inner rectangles**, and then average the outer rectangle and inner rectangle results (divide the sum of the two area by 2). Illustration of the inner rectangles is shown below:



In regard to areas for inner rectangles (the 5th one has height 0, so the 5^{th} rectangle area = 0):

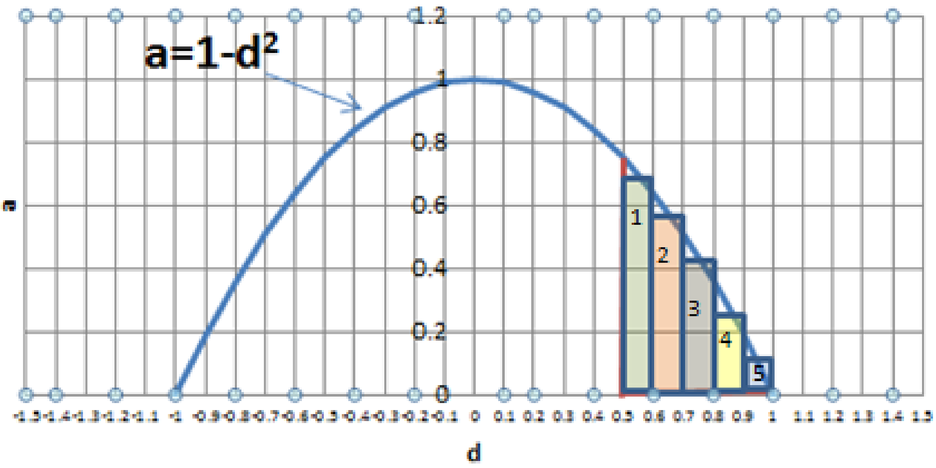
$$A = \sum_{\#=1}^{\#=5} A(\#) = (0.64)(0.1)+0.51(0.01)+0.36(0.1)+0.19(0.1)+0(0.1)$$

$$= 0.17 \text{ square units}$$



The average of the outer and inner rectangle areas is given by: (0.245+0.17)/2 = 0.2075.

Another method is to consider the mid-point rectangles on the function curve as illustrated below.



The area of the mid-point rectangles is determined as:

$$A = \sum_{\#=1}^{\#=5} A(\#) = (0.68)(0.1)+0.58(0.01)+0.44(0.1)+0.27(0.1)+0.97(0.1)$$

$$= 0.209 \text{ square units}$$

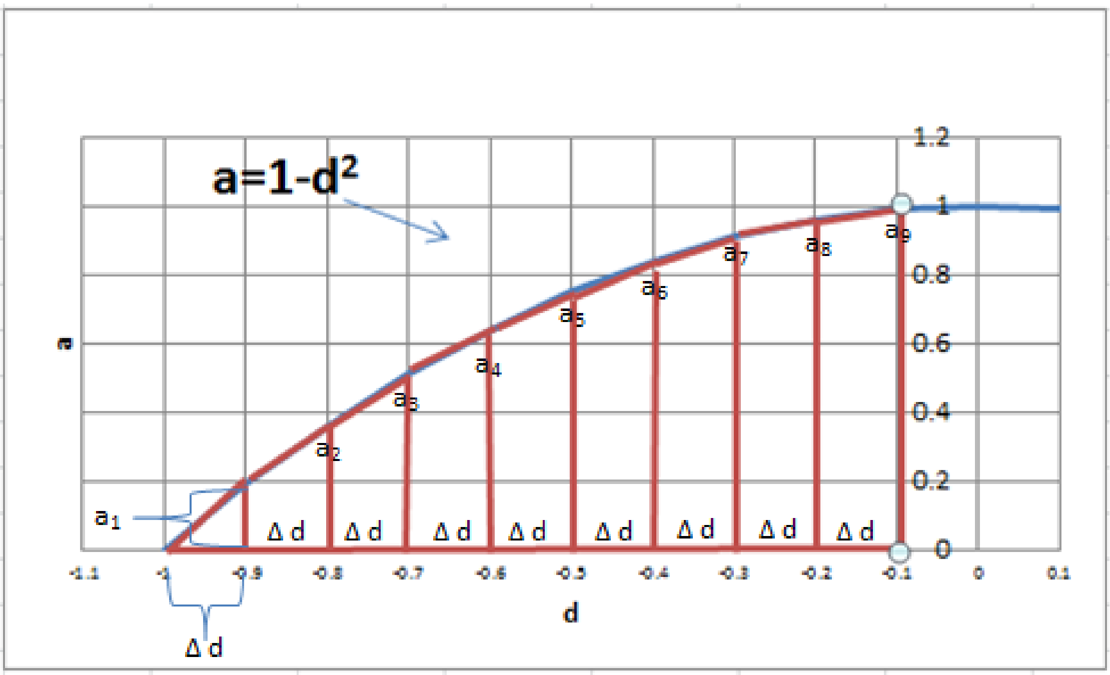
The Trapezoidal and Parabolic (Simpson's Rule) Methods

Two useful **numerical methods to the area under the curve are based on Trapazoidal and Parabolic calculations.** (Not unusual, computer software and graphic calculators tend to rely on these methods).

- Trapezoidal method
- Paraboli (Simpson's) method

The Trapezoidal Method

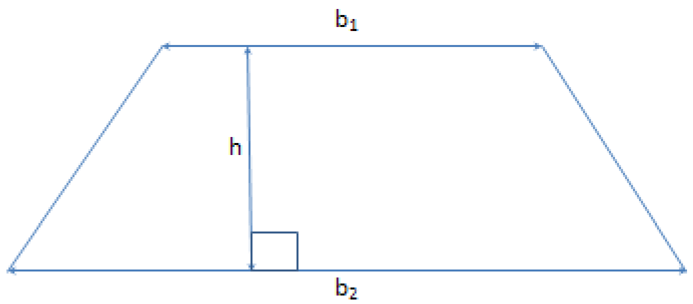
In place of rectangles, **trapezoids** (trapeziums) may be used that will give more accurate area under the curve estimate.



And as before, "Δd" represents a small incremental change in d.

Recall a trapezoid is a 4-sided flat shape with straight sides that has a pair of opposite sides parallel. (Called a trapezium in the UK. Both US and UK definitions of trapezium and trapezoid are swapped over.) The sides that are parallel are called "bases".

The area of a trapezoid (trapezium) is given by:



h = height or distance between the parallel bases.
b_1 and b_2 = length of each 'base' of the parallel legs

Area = $2h(b_1+b_2)$

Thus the estimated area under the curve of a trapezoid is found by adding the area of the trapezoids, similarly as for rectangles. (Trapezoids are generally vertical on graphs, or rotated 90°, thus their height, h, in this example is $h = \Delta d$.

Total Area of all Trapezoid $\approx 2(a_0+a_1)\Delta d+2(a_1+a_2)\Delta d+2(a_2+a_3)\Delta d+\ldots$, and so on.

Simplified… the **Trapezoidal Method,** for *n* trapezoids**:**

$$\text{Area} \approx \Delta d(2a_0+a_1+a_2+a_3+\ldots+2a_n)$$

Δd for each of the trapezoid areas from $d = d_1$ to $d = d_2$, is:

$\Delta d=(d_2-d_1)/n$

To determine the bases of each trapezoid (b_1 , $b_{2\ldots}b_n$), use the curve function or equation…

$a_0=f(0)$

$a_1=f(d+\Delta d)$

$a_2=f(d+2\Delta d)$

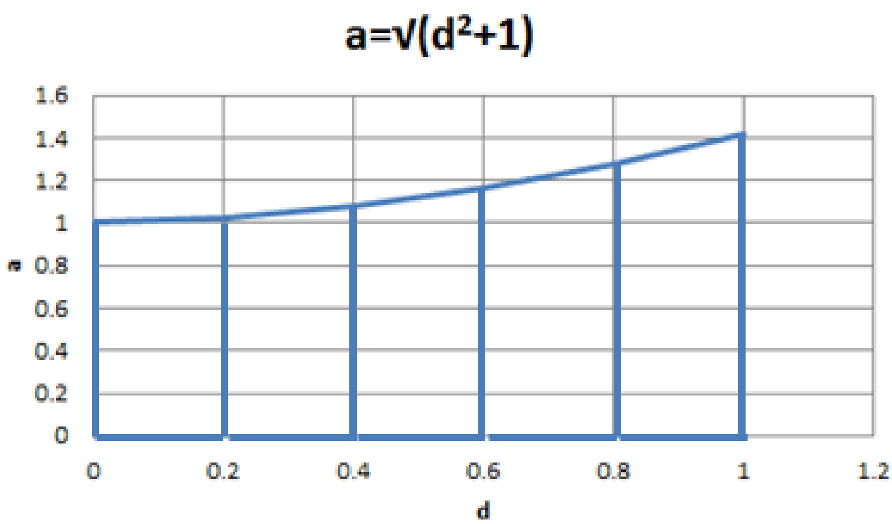
$a_n = f(\text{d})$

- As with rectangles, the more trapezoids used (but up to a practical limit), the more accurate the area estimate
- Consequently with more trapezoids, Δd approaches a limit of zero (0), expressed as, $\Delta d \rightarrow 0$, or the Limit $\Delta d_{\Delta d \rightarrow 0} = 0$.
- The area of all trapezoid if the function or curve bounding the area of interest, is above the *x (or d in this example)* -axis only between $x = d_0$ and $x = d_n$

A**r**ea Under the Curve = $\int_a^b f(d)d(d) \approx \Delta d(2a_0+a_1+\ldots+2a_n)$

Assuming n=5, approximate the area under the curve (integral) using the trapezoid method:

$$\int_0^1 \sqrt{(d^2+1)}\, d(d)$$



For this example; $a_0 = 0$ and $a_{n=5} = 1$, and the height or width of each trapezoid:

$\Delta d = d(d) = (a_5 - a_0)/n = 0.2$

$a_0 = f(d) = f(0) = \sqrt{(0^2+1)} = 1$

$a_1 = f(d+\Delta d) = \sqrt{(0.2^2+1)} = 1.0198$

$a_2 = f(d+2\Delta d) = f(0.4) = \sqrt{(0.4^2+1)} = 1.0770$

$a_3 = f(d+3\Delta d) = f(0.6) = 1.1661$

$a_4 = 1.2806$

$a_5 = 1.4142$

Estimated area under the trapezoidal method curve (integral) ≈

$0.2(2\times1+1.0198+1.0770+1.1661+1.2806+2\times1.4142)$

$= 1.15$

Or, expressing as an integral function: $\int_0^1 \sqrt{(d^2+1)}\, d(d) \approx 1.15$

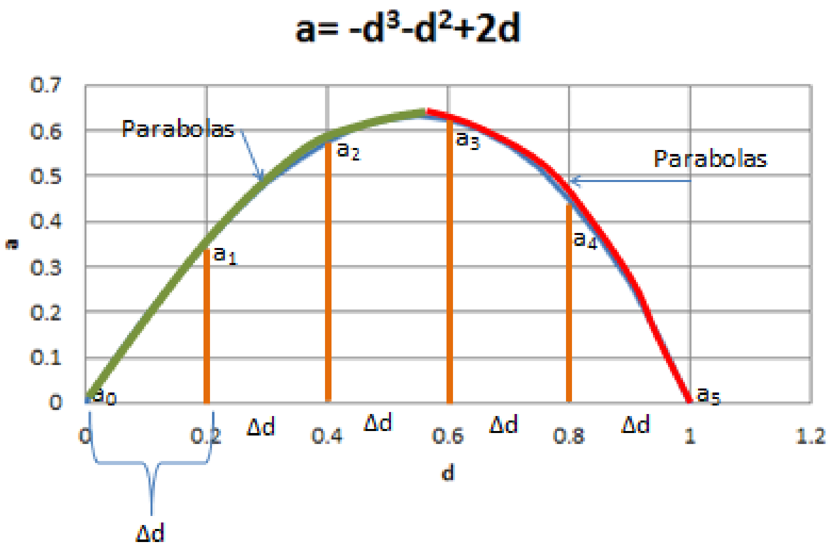
As noted in the above graph, the trapezoids follow closely the actual curve, thus the area estimate is closer to the actual area value when compared to the rectangle method (and the actual integral value is 1.148).

Parabolic Method or Simpson's Rule

As noted, the Trapezoid Method estimated area determinations that are closer to the actual area under the curve, in contrast to the rectangular method, primarily because less area outside the area under the curve, is used in the method.

Even so, there is a closer estimate method for determining area under a curve.

Using the Parabolic Method, or more conventionally called '**Simpson's Rule'**, **parabolas are used (in place of rectangles or trapezoids)** to estimate the area under the curve. Simpson's rule, as shown below, is more accurate than the other (rectangle or trapezoidal) numerical methodologies, because 'less' extraneous area outside the actual area under the curve, is used in the method.



R area under the curve is divided into n equal segments of width Δd. The estimated area is given by:

Simpson's Rule

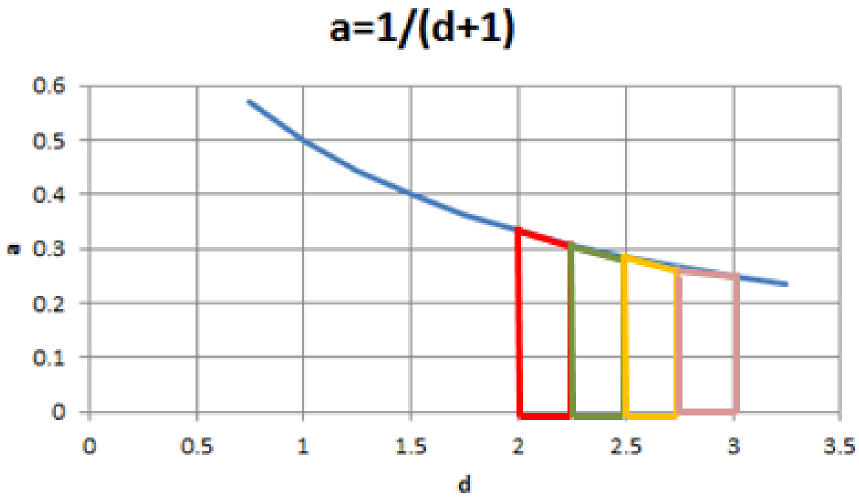
Area = $\int_{d_0}^{d_n} f(\mathrm{d})\,d(d) \approx (\Delta \mathrm{d}/3)\,(a_0+4a_1+2a_2+4a_3+2a_4\ldots+4a_{n-1}+a_n)$

where $\Delta \mathrm{d} = (d_n - d_0)/n$

In Simpson's Rule, *n* must be an even number.

Example of Simpson's Rule

Estimate $\int_2^3 1/(\mathrm{d}+1))\,d(d)$ using Simpson's Rule, assume n = 4.



$\Delta d = (d_4 - d_0)/n = (3\text{-}2)/4 = 0.25$

$a_0 = f(\mathrm{d}) = f(2) = 1/(2+1) = 0.3333$

$a_1 = f(\mathrm{d}+\Delta d) = f(2.25) = 1/(2.25+1) = 0.3076$

$a_2 = f(\mathrm{d}+2\Delta \mathrm{d})\mathrm{d} = f(2.5) = 1/(2.5+1) = 0.2857$

$a_3 = 0.2666$

$a_4 = 0.25$

S_o

Area Under the Parabolic Bounded Curves = $\int_{d_0}^{d_n} f(\mathrm{d})\mathrm{d}(\mathrm{d})$

$(0.25/3) * [0.3333+4(0.3076)+2(0.2857)+4(0.2666)+0.25]$

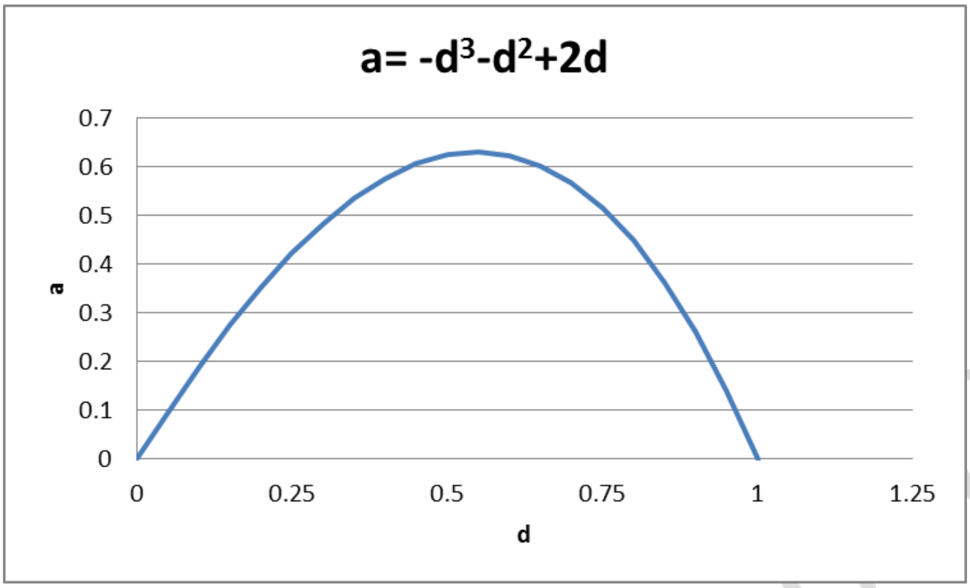
$= 0.2876$

The definite answer is 0.28766%.

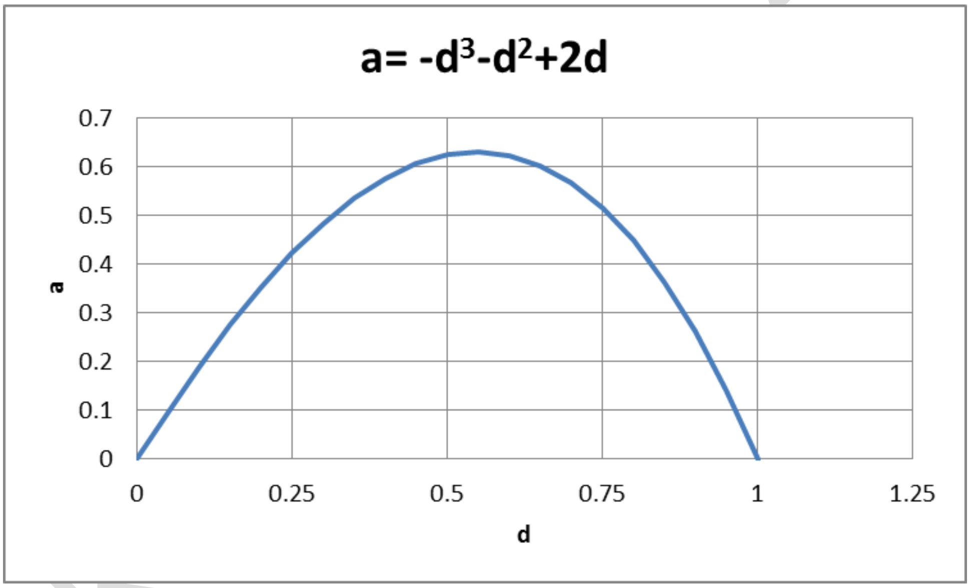
In this illustration, the actual curve is parabolic, so the overlapping area under the curve parabolas essentially *merge with the parabolic curve a=1/(d+1)*

Discussion of the Parabolic Method or Simpson's Rule

Consider finding the area under the curve…

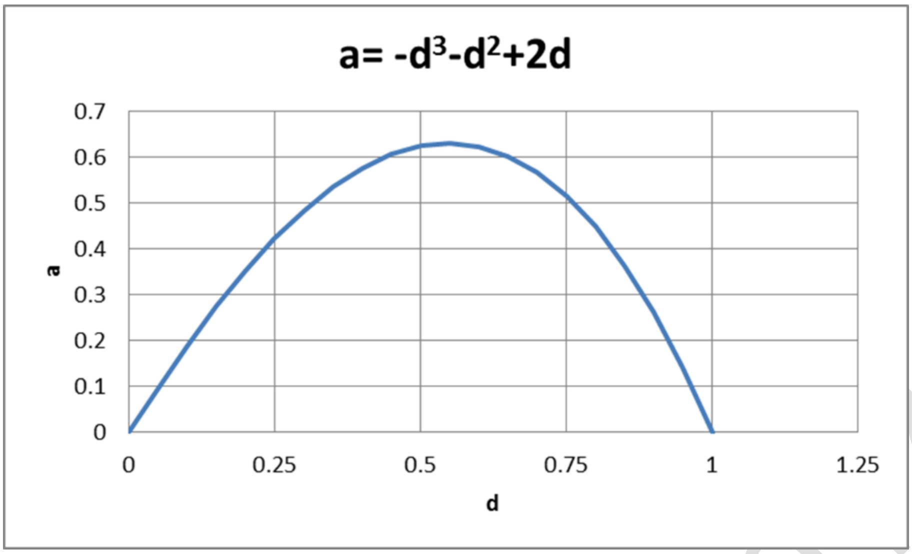


Divide the area under the curve into 4 equal sub-areas. (an even number)



Next construct parabolas which closely approximates the actual curve in each of the 4 sub-areas. Any three points, a specific parabola can mimic the actual sub-area boundary that overlaps the actual curve.

Let's start with the first 2 segments on the left. We take the end points, and the middle point as shown:



Measure each point using a overlaid grid, and determine these three points to be:

$(d_0,a_0) = (0,0)$

$(d_1,a_1) = (0.25,0.4219)$

$(d_2,a_2) = (0.5,0.625)$

With these three data points, consider the general form of a parabola, a $=ad^2+ bd + c$, and substitute the known d and a values…

$0 = a(0)^2+b(0)+c$

$0.4219 = a(0.25)^2+b(0.25) + c$

$0.625 = a(0.5)^2+b(0.5) + c$

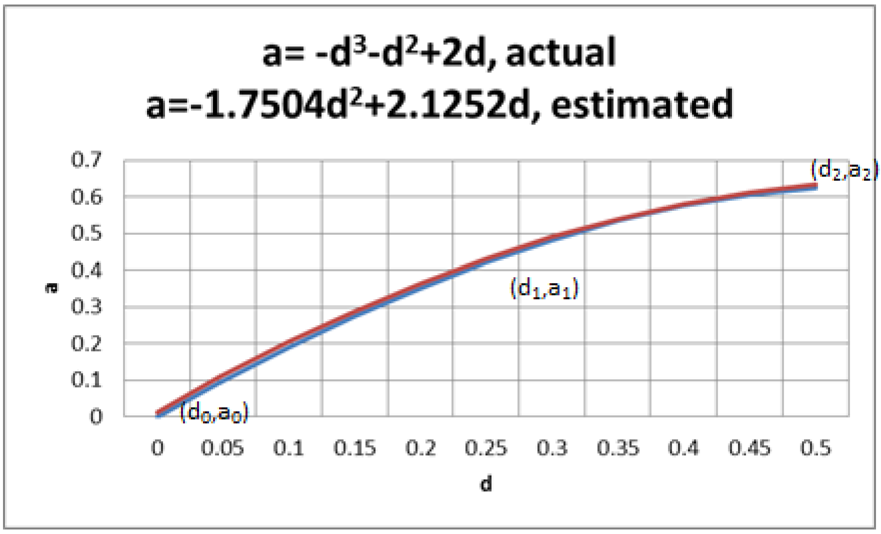
Thus three simultaneous equations in three unknowns. Solve with algebra…

a = -1.7504, b 2.1252, c=0.

The parabola passing through the three points is…

$a = -1.7504d^2 + 2.1252d + 0$

This parabolic equation in graphical form is…



As noted the estimated parabola passes through the three points, and it is close to our original curve, and thee more sub-areas chose, the more accurate.

A similar process is followed for the second half of the original curve.

Integral Calculus

Often the relationship involving **the rate of change** of two variables is known (such as speed or velocity, meters per second, the change in distance with respect to time), but a need to know the **direct** relationship between the two variables. For instance, the **velocity** of an object may be known at a particular time, but it is also needed the **position** of the object at a particular time.

To determine this direct relationship, the process called **integration** (or **antidifferentiation**)..***the Calculus… is considered.***

Uses of integration include finding centers of mass, displacement and velocity, fluid flow, modelling the behaviour of objects under stress, etc.

Before integration, the primary way to find the area under a curve was to draw rectangles with increasingly smaller widths.

Antiderivatives and The Indefinite Integral

"Antidifferentiation", also called "integration" is the **opposite process** to differentiation

Example 1

If the derivative is…

$dd/dt = 3xt^2$ (the rate of change of d with respect to t is determined as $3t^2$),

the origin of the function this derivative came from, then reverse or undo the differentiation process.

$d=t^3$ is one answer, since the derivative of $d = t^3 = dd/dt = 3t^2$.

There are many other antiderivatives that equally work…

$d=t^3+4$

$d=t^3+\pi$

$d=t^3+27.3$, since the derivative (or rate of change) of a constant value is zero (no rate of change of a constant).

In general, $d=t^3+C$ is the **indefinite integral** of $3t^2$. The number C is the **constant of integration (since the differential of a constant is zero)**.

The calculus integral expression is : $\int 3t^2 dt = t^3 + C$…

"The integral of $3t^2$ with respect to t equals $t^3 + C$."

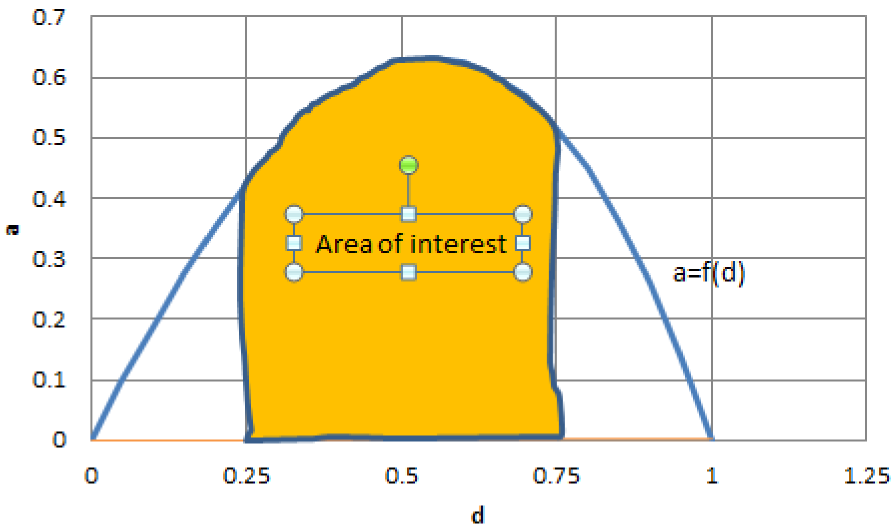
The ∫ sign is an elongated "S", standing for "sum". (In old German, and English, "s" was often written using this elongated shape.)

∑ is the symbol for "sum".

In calculus, ∫ is the symbol for the sum of an infinite number of infinitely small areas (or other variables).

Note: Sometimes a capital letter is used to signify integration. For example, F(t) to mean the integral of $f(t)$ or $\int$(f(t).

Integration was developed as a more efficient process than "adding areas of rectangles or trapezoids or parabolas".



By an incremental 'infinite sum' delta or, Δ-process, the *exact* area under a curve a = *f*(d) from $d=a_1$ (0.25 in the example) to $d=a_2$ (0.75 in the example) is given by the **definite integral:**

Area under the curve = $\int_{d_1}^{d_2} \mathbf{(f(d)d(d))}$

If *F*(*d*) is the integral (∫), of *f*(*d*), then

Area under the curve = $\int_{d_1}^{d_2} \mathbf{(f(d)d(d)} = \mathbf{[F(d)]}_{d_1}^{d_2} = \mathbf{F(d_2) - F(d_1)}$

To evaluate a **definite integral for the area under a curve**, proceed as follows:

- integrate the given function (be sure make an contant, C, zero)
  - Example integration for 3rd degree polynomial
  - $dy/dt = at^3+bt^2+ct+d$; where a,b,c,d are constants
  - $f(t) = dy/dt = at^n + bt^{n-1} + ct^{n-2} + \ldots + d$
  - $F(t) = \int dy/dt = \int [\, at^n + bt^{n-1} + ct^{n-2} + \ldots + d]$

- Integrating, $y = a(1/(n+1))t^{n+1} + b(1/(n-1+1))t^{n-1+1} + c(1/(n-2+1))t^{n-2+1} + \ldots + dt^{0+1}(1/(0+1))$
- For n = 3; $y = a(1/(3+1))t^{3+1} + b(1/(3-1+1))t^{3-1+1} + c(1/(3-2+1))t^{3-2+1} + dt^{0+1}(1/(0+1)) = a(1/(4))t^4 + b(1/(3))t^3 + c(1/(2))t^2 + dt(1/(1))$
- Check by reverse differentiation of the integral value: $at^3 + bt^2 + ct + d$ (check)

- substitute the **upper limit** (t_n) into the integral
- substitute the **lower limit** (t_1) into the integral
- subtract the second value from the first value
- the answer will be a **number representing the area under the curve.**

This is part of **The Fundamental Theorem of Calculus.**

Another practice…

Evaluate: $\int_1^5 3t^2 dt = [3/3t^3]_1^5$ *(Integrating…)*

$=[t^3]_1^5$

$=5^3-1^3$ (Substitute upper and lower values and subtract)

$=125-1 = 124$

The Definite Integral

As discussed previously

Area under the curve $= \int_{d_1}^{d_2} (f(d)d(d) = [F(d)]_{d_1}^{d_2} = F(d_2) - F(d_1)$

to find the area under a curve.

F(d) is the integral of f(d);

$F(d_2)$ is the value of the integral at the upper limit $d=db_2$; and

$F(d_1)$ is the value of the integral at the lower limit, $d=d_1$.

This expression is called a **definite integral.**

Practice…

Evaluate $\int_1^5 (3t^2+4t+1)dt$

1) Find the integral;write the upper and lower limits with square brackets:

$[(3/3)t^3+(4/2)t^2+t]_1^5$; $[t^3+2t^2+t]_1^5$

2) Substitute 5 (the upper limit) into the integral:

$[(5)^3+2(5)^2+5]=125+50+5=180$

3) Then substitute 1 into the integral:

$[(1)^3+2(1)^2+1]=1+2+1=4$

4) Subtract the result of (3) from the result of (2):

$180-4=176$

The integral expression is written as…

$\int_1^5 (3t^2+4t+1)dt=[t^3+2t^2+t]_1^5$

$[(5)^3+2(5)^2+5]-[(1)^3+2(1)^2+1]$

$=180-4$

$=176$

The answer is a **number** and does not involve "**+ C**", the constant of integration since the result is the **definite** integral.

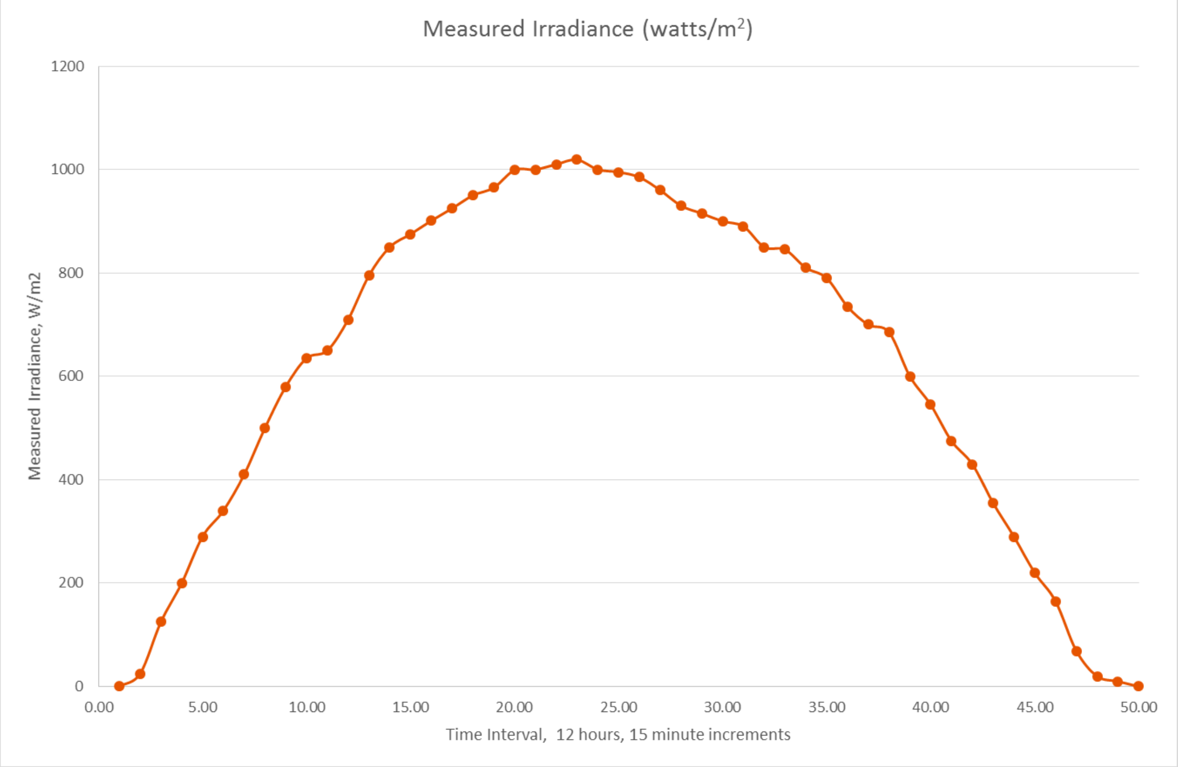
APPENDIX B

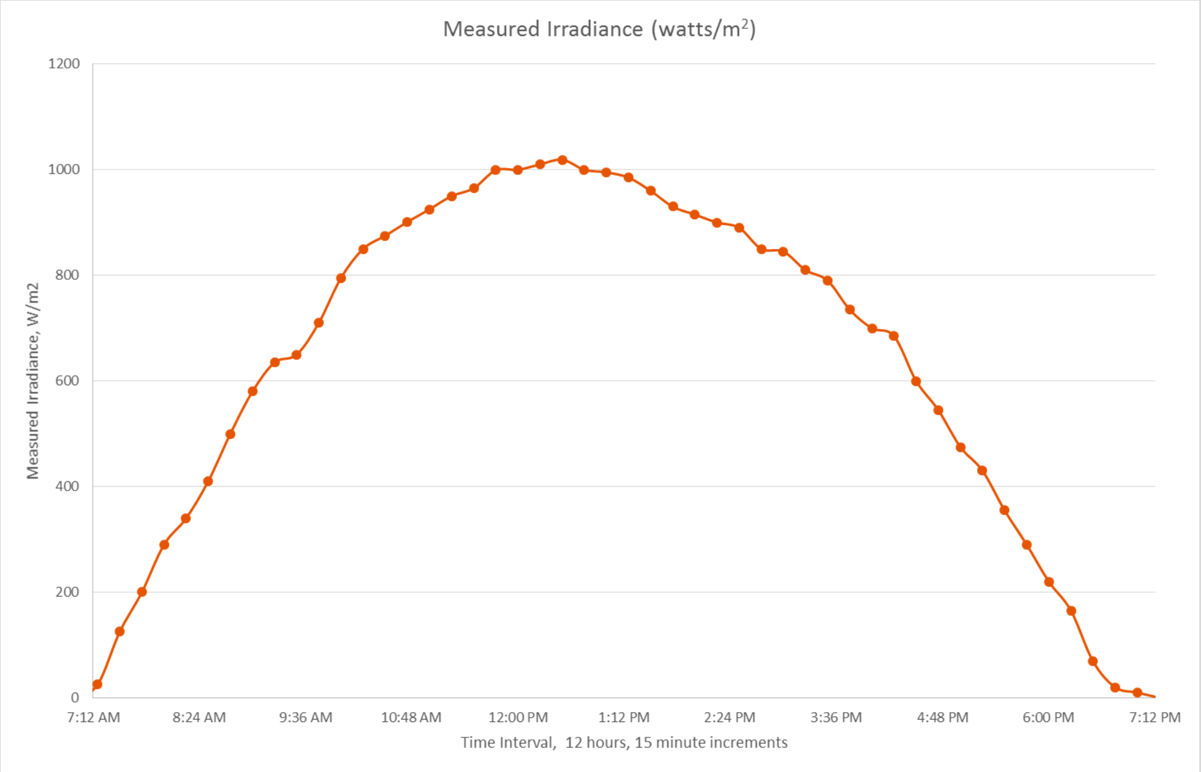
MEASURED IRRADIANCE DATA

Data Point		Time	Tiime Measurement Increment (minutes)	Measured Irradiance (watts/m^2)
1.00	Sunrise	7:00	0:00	0
2.00		7:15	0:15	25
3.00		7:30	0:15	125
4.00		7:45	0:15	200
5.00		8:00	0:15	290
6.00		8:15	0:15	340
7.00		8:30	0:15	410
8.00		8:45	0:15	500
9.00		9:00	0:15	580
10.00		9:15	0:15	635
11.00		9:30	0:15	650
12.00		9:45	0:15	710
13.00		10:00	0:15	795
14.00		10:15	0:15	850
15.00		10:30	0:15	875
16.00		10:45	0:15	901
17.00		11:00	0:15	925
18.00		11:15	0:15	950
19.00		11:30	0:15	965
20.00		11:45	0:15	999
21.00		12:00	0:15	1000
22.00		12:15	0:15	1010
23.00		12:30	0:15	1019
24.00		12:45	0:15	1000
25.00		13:00	0:15	995
26.00		13:15	0:15	985
27.00		13:30	0:15	960
28.00		13:45	0:15	930
29.00		14:00	0:15	915
30.00		14:15	0:15	900
31.00		14:30	0:15	890
32.00		14:45	0:15	850
33.00		15:00	0:15	845
34.00		15:15	0:15	810
35.00		15:30	0:15	790
36.00		15:45	0:15	735
37.00		16:00	0:15	700
38.00		16:15	0:15	685
39.00		16:30	0:15	600
40.00		16:45	0:15	545
41.00		17:00	0:15	475
42.00		17:15	0:15	430
43.00		17:30	0:15	355
44.00		17:45	0:15	290
45.00		18:00	0:15	220
46.00		18:15	0:15	165
47.00		18:30	0:15	69
48.00		18:45	0:15	20
49.00		19:00	0:15	10
50.00	Sunset	19:15	0:15	0

APPENDIX C

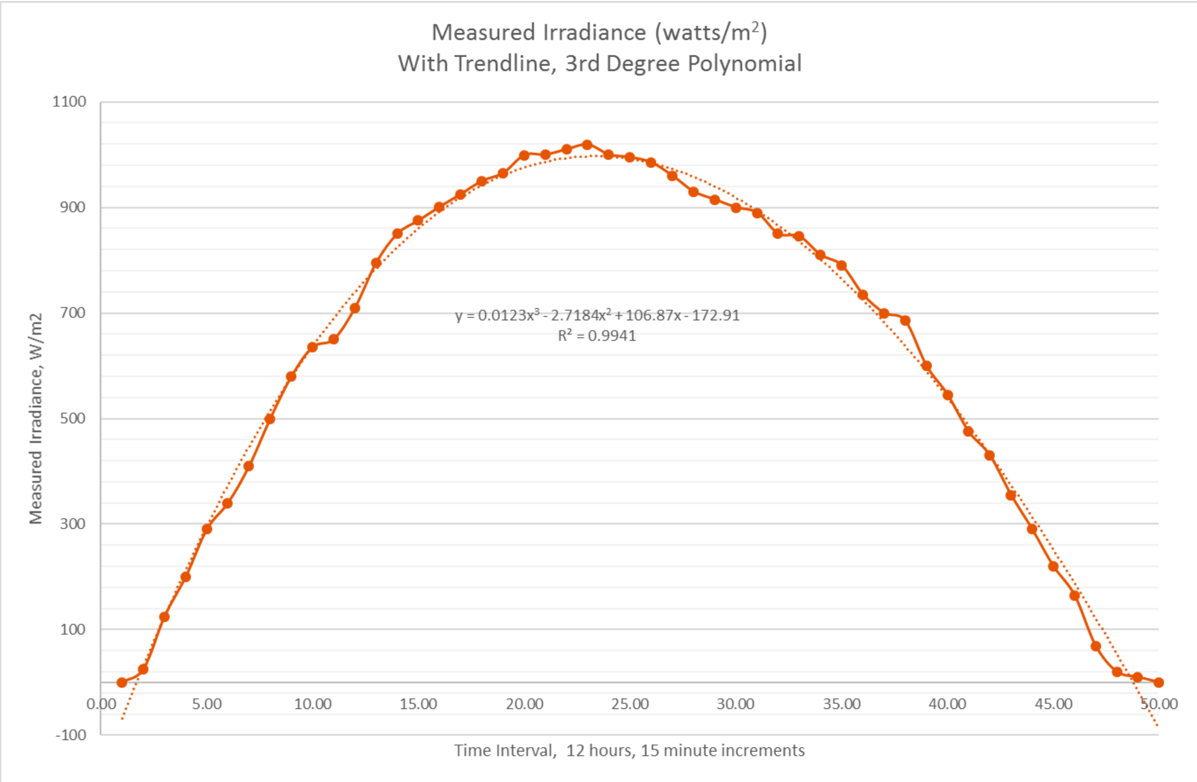
GRAPHS, MEASURED IRRADIANCE VS TIME

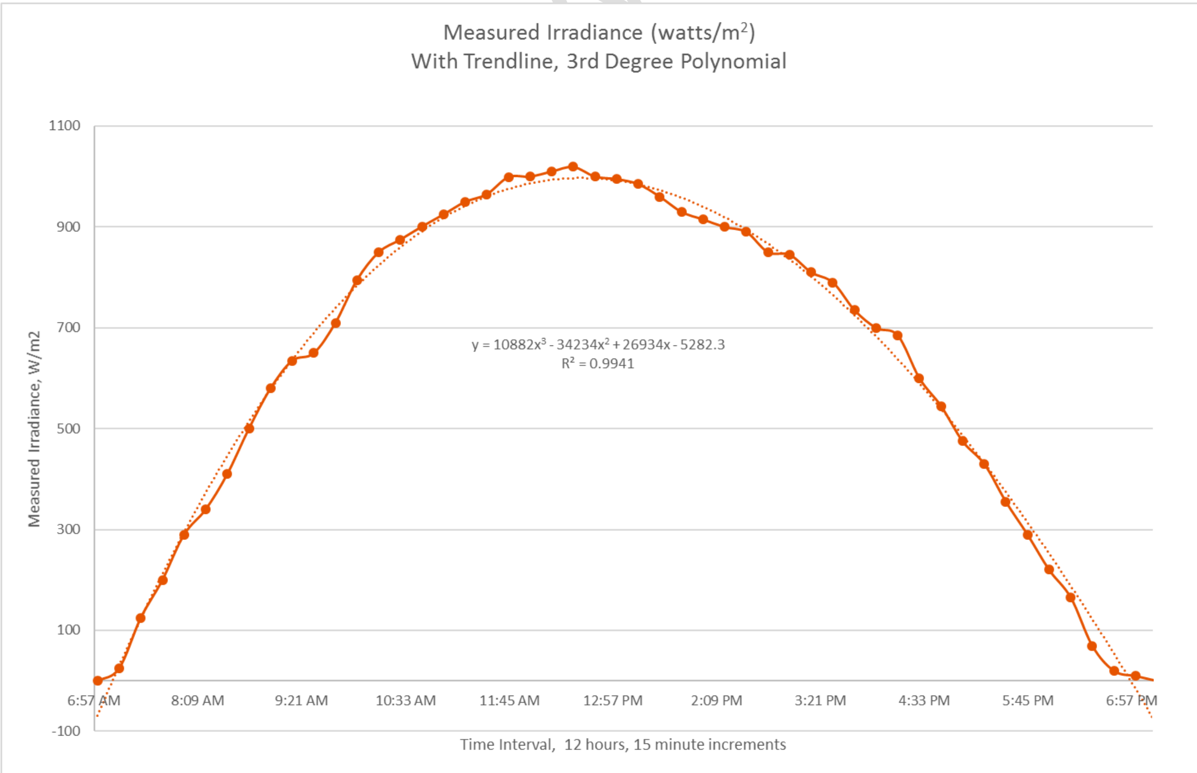




APPENDIX D

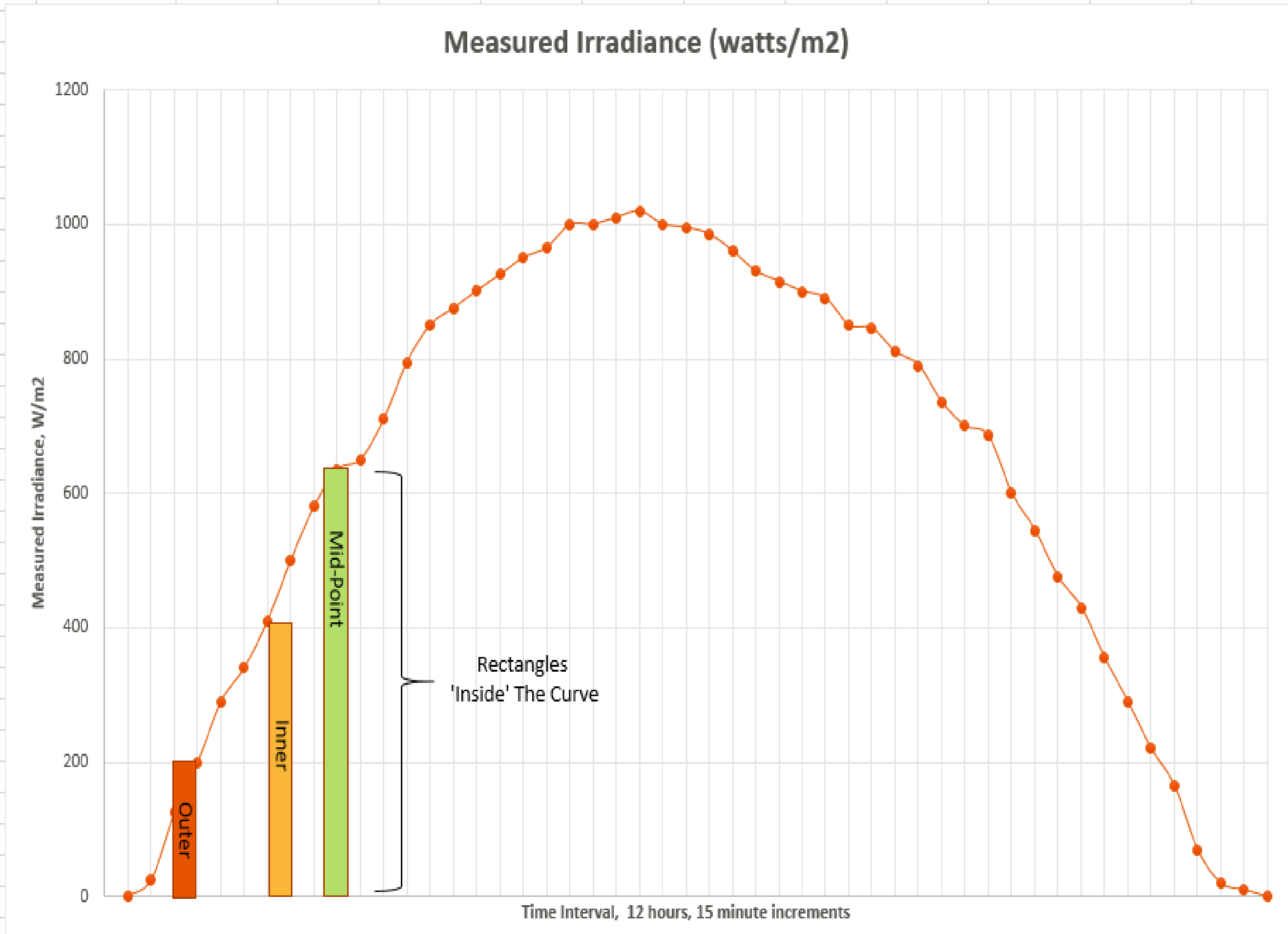
GRAPHS, MEASURED IRRADIANCE VS TIME WITH TRENDLINE EQUATIONS





APPENDIX E

SUM OF THE AREA RECTANGULES METHOD

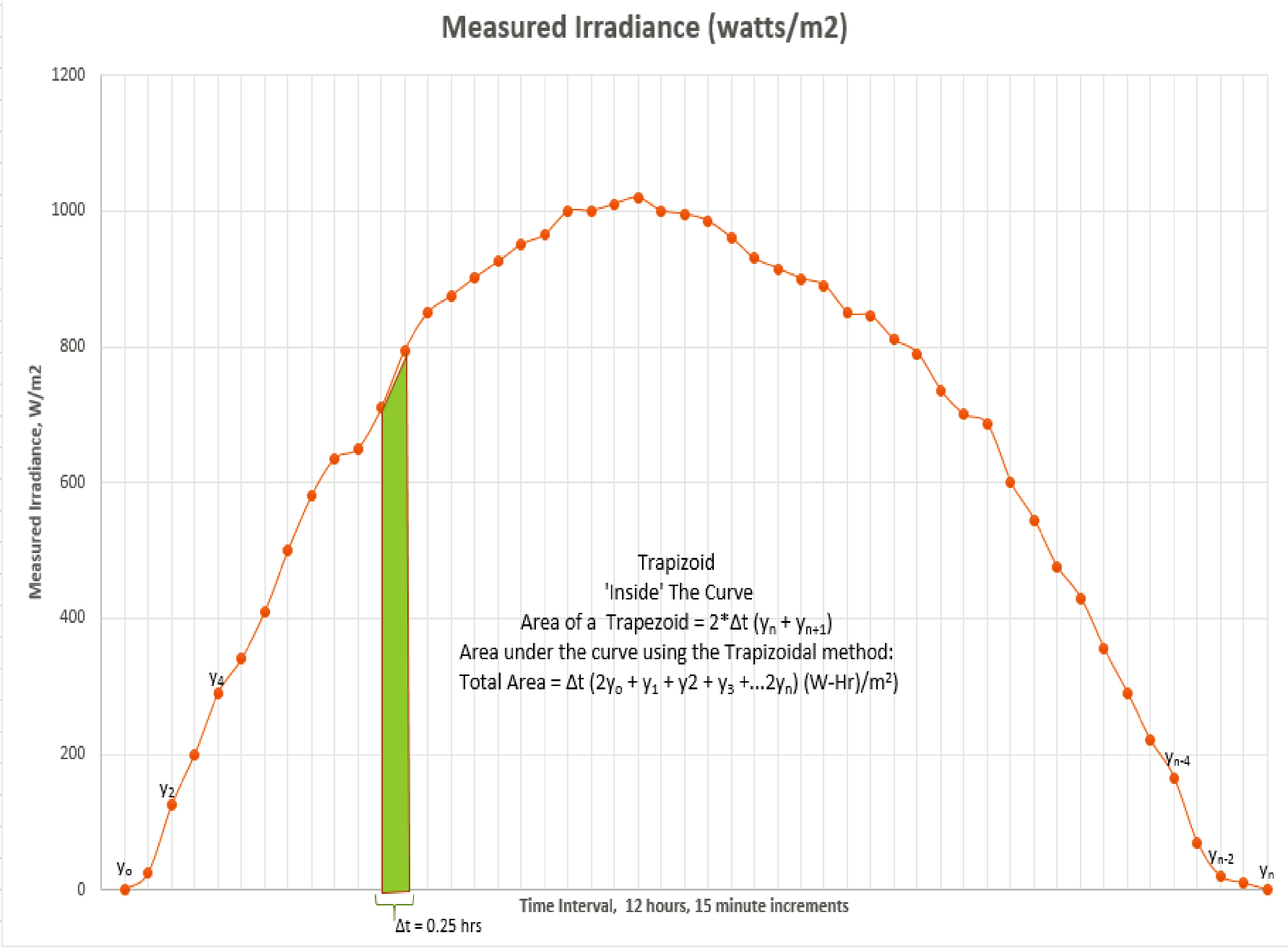


APPENDIX E, CONT'D

		(y) = Irradiance, W/m²			Area (Irradiance x Hours) (W-Hr)/m²		
Time Interval	Hours (x)	Outer (O)	Inner (I)	Mid-Point (MP)	Outer (x * O)	Inner (x * I)	Mid-Point (x * MP)
1.00	0.25	25	0	13	6.3	0.0	3.1
2.00	0.25	125	25	75	31.3	6.3	18.8
3.00	0.25	200	125	163	50.0	31.3	40.6
4.00	0.25	290	200	245	72.5	50.0	61.3
5.00	0.25	340	290	315	85.0	72.5	78.8
6.00	0.25	410	340	375	102.5	85.0	93.8
7.00	0.25	500	410	455	125.0	102.5	113.8
8.00	0.25	580	500	540	145.0	125.0	135.0
9.00	0.25	635	580	608	158.8	145.0	151.9
10.00	0.25	650	635	643	162.5	158.8	160.6
11.00	0.25	710	650	680	177.5	162.5	170.0
12.00	0.25	795	710	753	198.8	177.5	188.1
13.00	0.25	850	795	823	212.5	198.8	205.6
14.00	0.25	875	850	863	218.8	212.5	215.6
15.00	0.25	901	875	888	225.3	218.8	222.0
16.00	0.25	925	901	913	231.3	225.3	228.3
17.00	0.25	950	925	938	237.5	231.3	234.4
18.00	0.25	965	950	958	241.3	237.5	239.4
19.00	0.25	999	965	982	249.8	241.3	245.5
20.00	0.25	1000	999	1000	250.0	249.8	249.9
21.00	0.25	1010	1000	1005	252.5	250.0	251.3
22.00	0.25	1019	1010	1015	254.8	252.5	253.6
23.00	0.25	1000	1019	1010	250.0	254.8	252.4
24.00	0.25	995	1000	998	248.8	250.0	249.4
25.00	0.25	985	995	990	246.3	248.8	247.5
26.00	0.25	960	985	973	240.0	246.3	243.1
27.00	0.25	930	960	945	232.5	240.0	236.3
28.00	0.25	915	930	923	228.8	232.5	230.6
29.00	0.25	900	915	908	225.0	228.8	226.9
30.00	0.25	890	900	895	222.5	225.0	223.8
31.00	0.25	850	890	870	212.5	222.5	217.5
32.00	0.25	845	850	848	211.3	212.5	211.9
33.00	0.25	810	845	828	202.5	211.3	206.9
34.00	0.25	790	810	800	197.5	202.5	200.0
35.00	0.25	735	790	763	183.8	197.5	190.6
36.00	0.25	700	735	718	175.0	183.8	179.4
37.00	0.25	685	700	693	171.3	175.0	173.1
38.00	0.25	600	685	643	150.0	171.3	160.6
39.00	0.25	545	600	573	136.3	150.0	143.1
40.00	0.25	475	545	510	118.8	136.3	127.5
41.00	0.25	430	475	453	107.5	118.8	113.1
42.00	0.25	355	430	393	88.8	107.5	98.1
43.00	0.25	290	355	323	72.5	88.8	80.6
44.00	0.25	220	290	255	55.0	72.5	63.8
45.00	0.25	165	220	193	41.3	55.0	48.1
46.00	0.25	69	165	117	17.3	41.3	29.3
47.00	0.25	20	69	45	5.0	17.3	11.1
48.00	0.25	10	20	15	2.5	5.0	3.8
49.00	0.25	0	10	5	0.0	2.5	1.3
50.00	0.25	0	0	0	0.0	0.0	0.0
				TOTAL	7730.75	7730.75	7730.75

APPENDIX F

SUM OF THE AREA TRAPEZODIAL METHOD



APPENDIX F

SUM OF THE AREA TRAPEZODIAL METHOD, cont'd

Time Interval	Hours (x)	(y_n) = Irradiance, W/m^2	Area (Irradiance x Hours) (W-Hr)/m2	
1.00	0.25	0	0	$2*y_o = 2*(0) = 0$
2.00	0.25	25	25	$y_1 = 25$
3.00	0.25	125	125	y_2
4.00	0.25	200	200	y_3
5.00	0.25	290	290	y_4
6.00	0.25	340	340	y_5
7.00	0.25	410	410	y_6
8.00	0.25	500	500	y_7
9.00	0.25	580	580	y_8
10.00	0.25	635	635	y_9
11.00	0.25	650	650	y_{10}
12.00	0.25	710	710	y_{11}
13.00	0.25	795	795	y_{12}
14.00	0.25	850	850	y_{13}
15.00	0.25	875	875	y_{14}
16.00	0.25	901	901	y_{15}
17.00	0.25	925	925	y_{16}
18.00	0.25	950	950	y_{17}
19.00	0.25	965	965	y_{18}
20.00	0.25	999	999	y_{19}
21.00	0.25	1000	1000	y_{20}
22.00	0.25	1010	1010	y_{21}
23.00	0.25	1019	1019	y_{22}
24.00	0.25	1000	1000	y_{23}
25.00	0.25	995	995	y_{24}
26.00	0.25	985	985	y_{25}
27.00	0.25	960	960	y_{26}
28.00	0.25	930	930	y_{27}
29.00	0.25	915	915	y_{28}
30.00	0.25	900	900	y_{29}
31.00	0.25	890	890	y_{30}
32.00	0.25	850	850	y_{31}
33.00	0.25	845	845	y_{32}
34.00	0.25	810	810	y_{33}
35.00	0.25	790	790	y_{34}
36.00	0.25	735	735	y_{35}
37.00	0.25	700	700	y_{36}
38.00	0.25	685	685	y_{37}
39.00	0.25	600	600	y_{38}
40.00	0.25	545	545	y_{39}
41.00	0.25	475	475	y_{40}
42.00	0.25	430	430	y_{41}
43.00	0.25	355	355	y_{42}
44.00	0.25	290	290	y_{43}
45.00	0.25	220	220	y_{44}
46.00	0.25	165	165	y_{45}
47.00	0.25	69	69	y_{46}
48.00	0.25	20	20	$y_{47} = 20$
49.00	0.25	10	10	y_{48}
50.00	0.25	0	0	$2*y_{49} = 2*0=0$
		Sum	30923	
		x 0.25	7730.75	

APPENDIX G

SIMPSON'S RULE

Time Interval	Hours (x)	(yn) = Irradiance, parabolic prediction *	Simpson Calculaition	=	
1.00	0.25	-69	yo	-69	
2.00	0.25	30	y1 *4	120	
3.00	0.25	124	y2*2	247	
4.00	0.25	212	y3*4	847	
5.00	0.25	295	y4*2	590	
6.00	0.25	373	y5*4	1492	
7.00	0.25	446	y6*2	892	
8.00	0.25	514	y7*4	2057	
9.00	0.25	578	y8*2	1155	
10.00	0.25	636	y9*4	2545	
11.00	0.25	690	y10*2	1380	
12.00	0.25	739	y11*4	2957	
13.00	0.25	784	y12*2	1568	
14.00	0.25	824	y13*4	3297	
15.00	0.25	860	y14*2	1720	
16.00	0.25	891	y15*4	3566	
17.00	0.25	919	y16*2	1837	
18.00	0.25	942	y17*4	3767	
19.00	0.25	961	y18*2	1921	
20.00	0.25	976	y19*4	3902	
21.00	0.25	986	y20*2	1973	
22.00	0.25	993	y21*4	3974	
23.00	0.25	997	y22*2	1993	
24.00	0.25	996	y23*4	3985	
25.00	0.25	992	y24*2	1984	
26.00	0.25	984	y25*4	3937	
27.00	0.25	973	y26*2	1946	
28.00	0.25	958	y27*4	3833	
29.00	0.25	940	y28*2	1880	
30.00	0.25	919	y29*4	3675	
31.00	0.25	894	y30*2	1788	
32.00	0.25	866	y31*4	3465	
33.00	0.25	835	y32*2	1671	
34.00	0.25	802	y33*4	3207	
35.00	0.25	765	y34*2	1530	
36.00	0.25	725	y35*4	2901	
37.00	0.25	683	y36*2	1366	
38.00	0.25	638	y37*4	2551	
39.00	0.25	590	y38*2	1180	
40.00	0.25	540	y39*4	2159	
41.00	0.25	487	y40*2	974	
42.00	0.25	432	y41*4	1727	
43.00	0.25	374	y42*2	748	
44.00	0.25	314	y43*4	1257	
45.00	0.25	252	y44*2	505	
46.00	0.25	188	y45*4	753	
47.00	0.25	122	y46*2	244	
48.00	0.25	54	y47 *4	216	
49.00	0.25	-16	y48	-32	
50.00	0.25	-88	0		
		Sum		93183	
		x(.25/3)		7765.237	W-Hr/m^2
*y = 0.0123x3 - 2.7184x2 + 106.87x - 172.91					