ANGULAR MOMENTUM AND MOMENT OF INERTIA

Contents

INTRODUCTION

A discussion of kinematic linear (translational) and rotational motion (of single particles and rigid bodies comprised of many particles) showing proofs of physics concepts along with diagrams and examples.

Topics include:

- Vectors, unit vectors, scalars, vector multiplication (dot product and cross product);
- Relationship between linear translational and rotational kinematics;
- Linear tangential velocity, radial centripetal and angular acceleration, momentum, inertia and kinetic energy;
- Angular speed, angular acceleration, torque, moment of inertia, angular momentum and kinetic energy.

In this article, a vector (an event with both magnitude and direction), such as traveling east (the direction) at 10 mph (the magnitude), is shown in bold letters, and for this velocity vector example, **v** = East 10 mph. When just the magnitude is relevant and not the direction of a vector or scalar, it is noted with vertical bars and not in bold type, example: |v|, which represents in this example 10 mph without reference to direction.

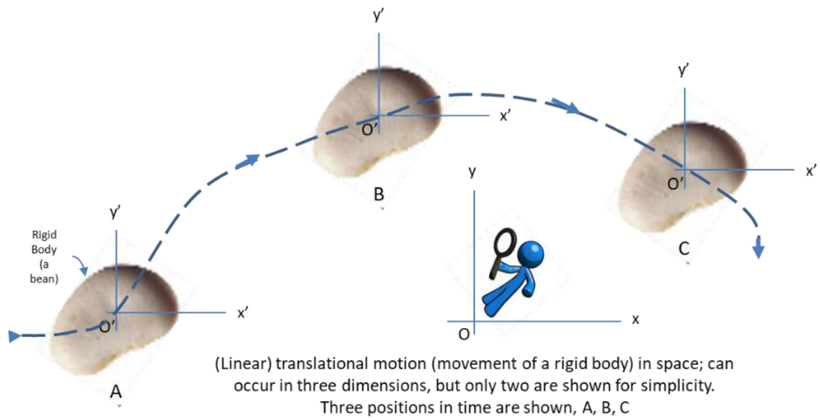
1. FUNDAMENTALS OF KINEMATIC LINEAR AND ROTATIONAL MOTION.

A refresher on **angular momentum** and **moment of inertia** since both are important physic concepts used in particle physics. But first some fundamentals of kinematic linear and rotational motion.

When objects (particles, molecules, stuff…) rotate, their movement known as rotational kinematics, is not linear and so the physics of such system must take into account rotational movement, which is more complex than linear or straight line movement.

Most kinematic (linear or rotational) deal with translational motion of single particles or of rigid bodies. No real body is truly rigid, but many bodies, such as molecules, steel beams, and planets, are rigid enough so that we can ignore the fact that they warp, bend or vibrate.

Illustrated in the below Figure, as a rigid body moves in pure translational of each particle of the body undergoes the same displacement (in space) as every other particle in any given time interval.

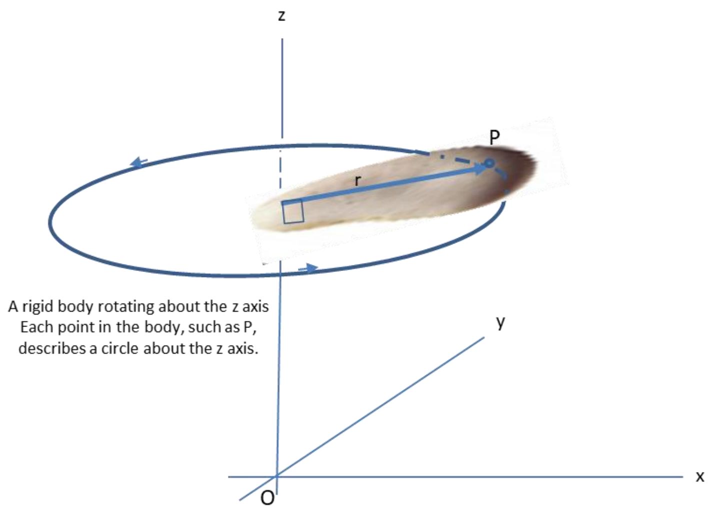


(Linear) translational motion (movement of a rigid body) in space; can occur in three dimensions, but only two are shown for simplicity. Three positions in time are shown, A, B, C

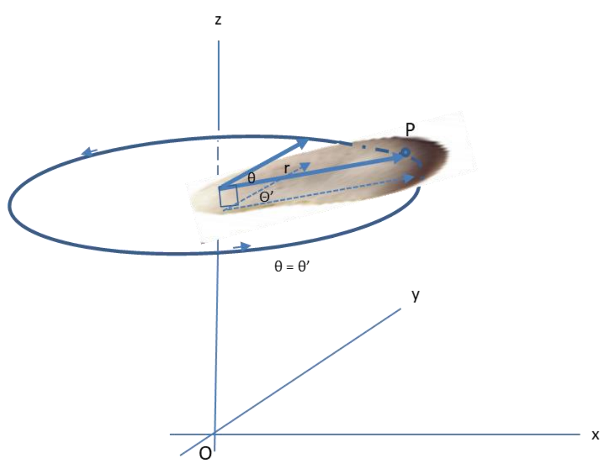
An observer (in reference frame x, y and z) will call motion of the rigid body *translational* if the axes of a reference frame which is imagined rigidly attached to the rigid body object, in this example say x′, y′ and z′, always remain parallel to the axes of the observer's reference frame, x, y and z, where x is parallel to x′, y || y′, z || z′. The Figure shows the translational motion of a rigid body moving from positions A, B and C (with respect to x, y and z). The path may be other than a straight line. Throughout the motion every point of the rigid body undergoes the same displacements as every other point. Thus it can be assumed that the rigid body to be a 'single' unit particle because in describing the motion of one point on the rigid body we have described the motion of the body as a whole.

In this discussion we will be interested in the rotation of the rigid body. For this discussion, we restrict ourselves to inelastic single particles and to rigid bodies, which means we will not consider such rotational motions as the elastic solar system or of water in a spinning beaker. We shall also deal only with rotation about axes that remain fixed in the reference frame in which we observe the rotation.

The below figure shows the rotational motion of a rigid body about a fixed axis z, of our reference frame. Let P represent a particle in the rigid body, arbitrarily selected and described by the position vector r. (where r has a defined direction and magnitude – and in this case being the radius of the subscribed circle P is describing as it rotates around the z axis). ***A rigid body moves in pure rotation if every particle of the body (such as P) moves in a circle, the centers of which are on a straight line called the axis of rotation (the z axis in this example)***.



If a perpendicular line is drawn from any point in the body to the axis, each such line will sweep through the same angle in any given time interval as another such line (see below figure).

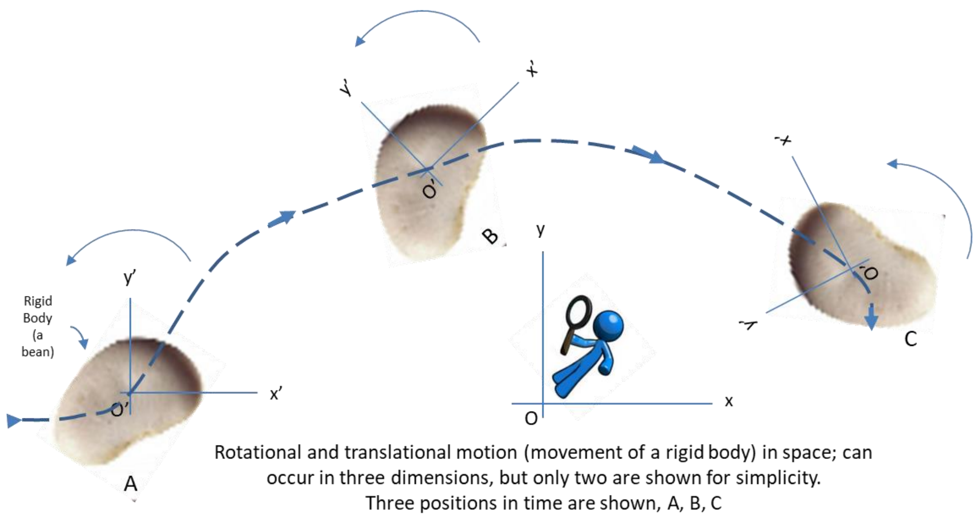


Thus we can describe the pure rotation of a rigid body by considering the motion of any one of the particles (such as P) that make it up, ruling out points directly on the axis of rotation since there is no vector magnitude or direction on such points).

The general motion of a rigid body is however, a combination of translation and rotation rather than one of pure rotation.

We can locate a rigid body that is moving in pure translation (no rotation) by giving the three coordinates x, y and z of any point (its center of mass , for example) in a particular reference frame.

In contrast, for a rigid body that rotates as it moves translationally (such as a thrown spinning frisbee), we need in the most general case, three more coordinates, such as angles, to specify the orientation of the body with respect to the reference frame. The below figure illustrates a special case of a rigid body motion combining translation and rotation.

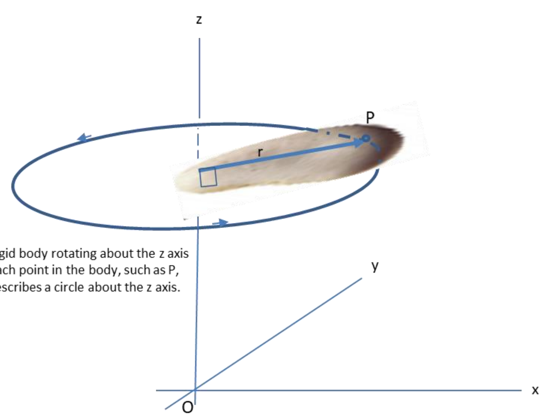


Rotational and translational motion (movement of a rigid body) in space; can occur in three dimensions, but only two are shown for simplicity.
Three positions in time are shown, A, B, C
The reference frame fixed on the body (x',y') changes its orientation with respect to x, y as the motion proceeds.

To locate the body we must not only locate point O in the body in the xy reference frame but we must also say how the x'y' reference frame, which is fixed in the body, is oriented with respect to the xy frame.

We can describe the translational motion of any system of particles – whether rigid or not – whether rotating or not – by imagining that all of the mass M of the body is concentrated at the center of mass and the F_{ext}, the resultant of the external forces acting on the body, acts at this point. The acceleration of the center of mass is then give by $F_{ext} = Ma_{cm}$. It is very to be able to represent the translational motion of a rigid body by the motion of a single point – its center of mass; all that is left is to determine its rotational motion. We shall discuss such combined translational and rotational motions later. This will be simpler to do after we have studied pure rotation about a fixed axis.

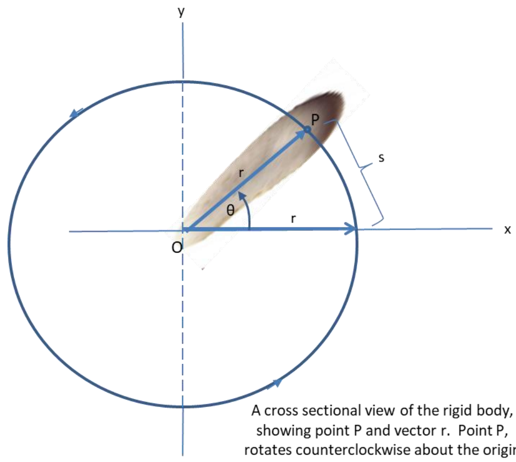
Now to describe the rotational motion of an object shown in the below figure.



A rigid body rotating about the z axis
Each point in the body, such as P, describes a circle about the z axis.

Such description is called rotational kinematics; we must define the variables of angular motion and relate them to each other, just as in particle kinematics, there is defined the variables of translational motion and related them to each other. Now to discuss rotational dynamics.

Noting the above figure, pass a plane through P at right angles to the axis of rotation (the z axis in this example) (see the below figure of such a plane looking downward on it from above, along the z axis).

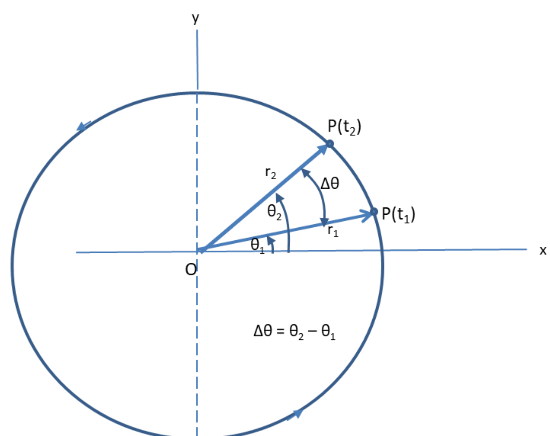


A cross sectional view of the rigid body, showing point P and vector r. Point P, rotates counterclockwise about the origin In a circle of radius r.

We can determine exactly where the entire rotating body is in our reference frame if we know the location of any single particle (P) of the body in this frame. Thus, for the kinematics of this example, we need only consider the (two-dimensional) motion of a particle in a circle. The angle θ is the angular position of particle P with respect to the reference position. We arbitrarily select the positive sense of rotation in this figure to be counterclockwise, so that θ increases for counterclockwise rotation and decreases (negative, - θ) for clockwise rotation.

It is convenient to measure θ in radians[1] rather in degrees.

By letting the above Figure body rotate counterclockwise, at time t_1 the angular position of P is θ_1 and at a later time t_2 its angular position is θ_2, as illustrated in the below figure.



The reference line r : radius (= OP), fixed in the rigid body is displaced through angle +Δθ (= $\theta_2 - \theta_1$) in time +Δt (= $t_2 - t_1$) as P rotates counterclockwise

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[1] The radian is a pure number, having no physical dimension since it is the ratio of two lengths. By definition θ is given in radians by the relation: θ = s/r, in which s is the arc length on the circle of the chord described by the angle θ and r is the radius of the circle. Since the circumference of a circle of radius r is (s=) 2πr, there are 2π radians in a complete circle, θ = 2πr/r = 2π, Therefore 2π radians = 360°; π radians = 180°; and 1 radian ≈ 57.3° (180°/ π).

The outline of the rigid body has been deleted for simplicity. The angular displacement of P will be $\Delta\theta = \theta_2 - \theta_1$ during the time interval of $\Delta t = t_2 - t_1$. The ***average angular speed*** ϖ (rotations per unit of time) $= (\theta_2 - \theta_1)/(t_2 - t_1) = \Delta\theta/\Delta t$, or change in radians per unit of time (typically radians per second, rad/sec). The ***instantaneous angular speed ω*** is defined as the limit approached by this ratio as Δt approaches zero: $\omega = \lim_{\Delta t \to 0} \Delta\theta/\Delta t = d\theta/dt$.

For a rigid body all radial lines fixed in it perpendicular to the axis of rotation rotate through the same angle in the same time, so that the angular speed ω about this axis is the same for each particle in the body. Thus ω is characteristic of the body as a whole. Angular speed has the dimensions of an invers time (T^{-1}); it unis are commonly taken to be radians/sec or rev/sec (revolutions per second).

Angular speed ω should not be confused with translational tangential velocity, v, which is the velocity in length per unit of time of P at the tangent point of P on the circle.

If the angular speed ω is not constant, then the particle has an angular acceleration (recall change in speed is acceleration). Let ω_1 and ω_2 be the instantaneous angular speeds (rad/sec) at the time t_1 and t_2 respectively; then the ***average angular acceleration*** $\bar{\alpha}$ of the particle P is defined as:

$\bar{\alpha} = (\omega_2 - \omega_1)/(t_2 - t_1) = \Delta\omega/\Delta t$; and the instantaneous angular acceleration is the limit of this ratio as Δt approaches zero or $\alpha = \lim_{\Delta t \to 0} \Delta\omega/\Delta t = d\omega/dt$.

Because angular speed ω is the same for all particles in the rigid body, it follows that α must be the same for each particle and thus α, like ω, is a characteristic of the body as a whole. Angular acceleration has the dimensions of an inverse time squared (T^{-2}); its units are commonly taken to be radians/sec^2 or rev/sec^2. The rotation of a particle (or a rigid body) ***about a fixed axis*** has a formal correspondence to the translational motion of a particle (or a rigid body) ***along a fixed direction***.

The kinematic variables for a rotating system are θ, ω, and α, and x, v, and a for a non-rotating translational system. Note the angular quantities differ dimensionally from the corresponding linear quantities by a length factor.

For translational motion of a particle or a rigid body along a fixed direction (non-rotational), such as the x-axis,

- the simplest type of motion is that in which the acceleration (a) is zero.
- the next simplest type corresponds to (a) = a constant (other than zero) and for this motion the equations in the below table have been derived, which connect the kinematic variable x, v, a and t (time) in all possible combinations.

KINEMATIC EQUATIONS FOR STRAIGHT LINE MOTION WITH CONSTANT ACCELERATION				
INITIAL CONDITIONS: t = 0; x_o position and v_{xo} velocity				
$\Delta t = (t_1 - t_2)$	Contains			
Equation	x	v_x	a_x	t
$v_x = v_{xo} + a_x\Delta t$	X	√	√	√
$x = x_o + ½[v_{xo} + v_x]\ \Delta t$	√	√	X	√
$x = x_o + v_{xo}\Delta t + ½[a_x t^2]$	√	X	√	√
$v_x^2 = v_{xo}^2 + 2a_x(x - x_o)$	√	√	√	X

For the rotational motion of a particle or a rigid body around a fixed axis

- the simplest type of motion is that in which the angular acceleration α is zero (such as uniform circular motion).
- The next simplest type of motion, in which α = a constant (other than zero), corresponds the same as linear motion with (a) = a constant (other than zero).

The below table lists the kinematic equations for motion for both constant linear and angular acceleration. Both sets of equations hold for particles and rigid bodies.

MOTION WITH CONSTANT LINEAR (a) AND ANGULAR (α) ACCELERATION $\Delta t = (t_1 - t_2)$	
TRANSLATIONAL MOTION (FIXED DIRECTION)	ROTATIONAL MOTION (FIXED AXIS)
$v_x = v_{xo} + a_x\Delta t$	$\omega = \omega_o + \alpha\Delta t$
$x = x_o + ½[v_{xo} + v_x)t$	$\theta = ½(\omega_o + \omega)\, \Delta t + \theta_o$
$x = x_o + v_{xo}\Delta t + ½[a_x\Delta t^2]$	$\theta = \omega_o\Delta t + ½(\alpha\Delta t^2) + \theta_o$
$v_x^2 = v_{xo}^2 + 2a_x(x - x_o)$	$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$

For the angular quantities, we arbitrarily select one of the two possible directions of rotation about the fixed axis as the direction which θ is increasing. Thus for $\omega = d\,\theta\,/dt$, if θ is increasing in time then ω is positive. Similarly, for $\alpha = d\omega/dt$ if ω is increasing with time α is positive.

Translational and rotational variables and their unit conventions are illustrated as follows:

x (length)	Unit length	θ (radial distance)	radians
v (velocity)	Length/time, mph	ω (angular speed)	Rad/sec; Rev/sec
a (acceleration)	Length/time^2, ft/sec^2	α (angular acceleration)	Rad/sec^2; Rev/sec^2

Linear displacement x, velocity v and acceleration a are vectors. They have direction and magnitude. ***The corresponding angular quantities may or may not be vectors***, for in addition to a magnitude we must also specify a direction for them, namely the direction of the axis of rotation in space. Because we considered rotation only out of a fixed axis, we were able to treat θ, ω and α as scalar quantities (having magnitude only and no bearing on direction). HOWEVER, ***if the direction of the axis changes, we can no longer avoid the question "are rotational quantities vectors"? We can find out only by seeing whether or not they obey the laws of vector addition.***

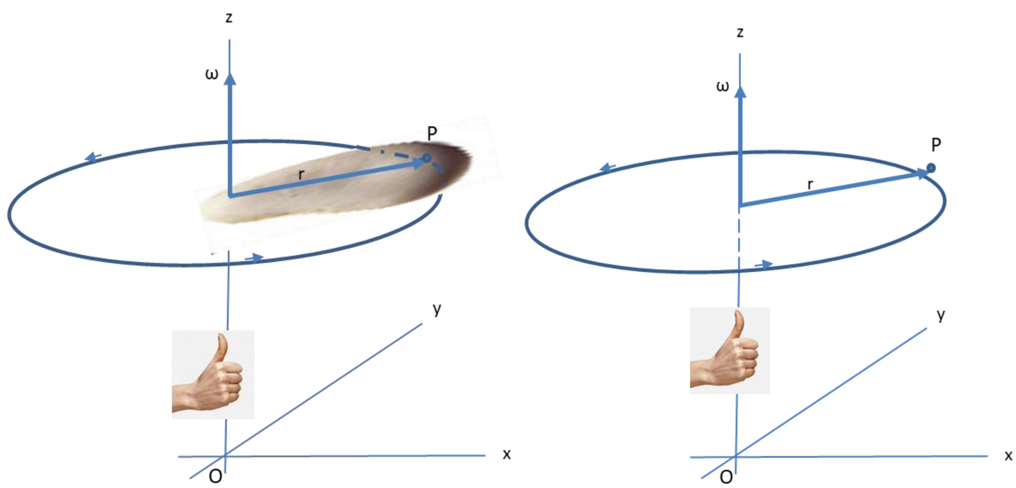
ARE ANGULAR QUANTITIES VECTORS?

Considering the angular displacement θ – may or may not be a vector: is not a vector if finite magnitude (NOT A VECTOR IF BIG) but is if infinitesimal angular displacement (IS A VECTOR IF SMALL) . The magnitude of the angular displacement of a body is the angle through which the body turns. Angular displacements, however are NOT vectors because they do NOT add like vectors. For example, given two successive rotations θ_1 and θ_2 to a book which initially lies flat on a table. Let rotation θ_1 be a 90^o clockwise turn about a vertical axis through the center of the book as we view it from above. Let θ_2 be a 90^o clockwise turn about a north-south axis through the center of the book as we view it looking north. In one case, apply operation θ_1 first then θ_2. In the other case, apply operation θ_2 first then θ_1. If angular displacements are vector quantities, they must add like vectors. In particular, they must obey the law of vector addition $\theta_1 + \theta_2 = \theta_2 + \theta_1$, which tells us that the order in which we add vectors does not affect their sum. This law fails for finite angular displacements. **Hence finite (big) angular displacements are not vector quantities.**

Suppose that instead of 90° rotations we had made 3° rotations. The result of $\theta_1 + \theta_2$ would still differ from the result of $\theta_2 + \theta_1$, but the difference would be much smaller. As the angular displacements are made smaller, the difference between the two sums disappears rapidly. If the angular displacements are made infinitesimal, the order of addition no longer affects the results. Hence, ***infinitesimal angular displacement are vectors.***

In contrast, linear displacement of x_1 and x_2 which is a vector, is indifferent which x is selected first, since the resultant vector will be the same no matter which order the linear displacement is selected.

Considering angular velocity of speed – is a vector. Quantities defined in terms of infinitesimal angular displacements may themselves be vectors. Since angular velocity (speed) is $\omega = d\theta/dt$ which means the derivative of angular displacement $d\theta$ is approaching an (infinitesimal) limit, which means $d\theta$ is a vector (as above) and dt a scalar, then the quotient ω is a vector.

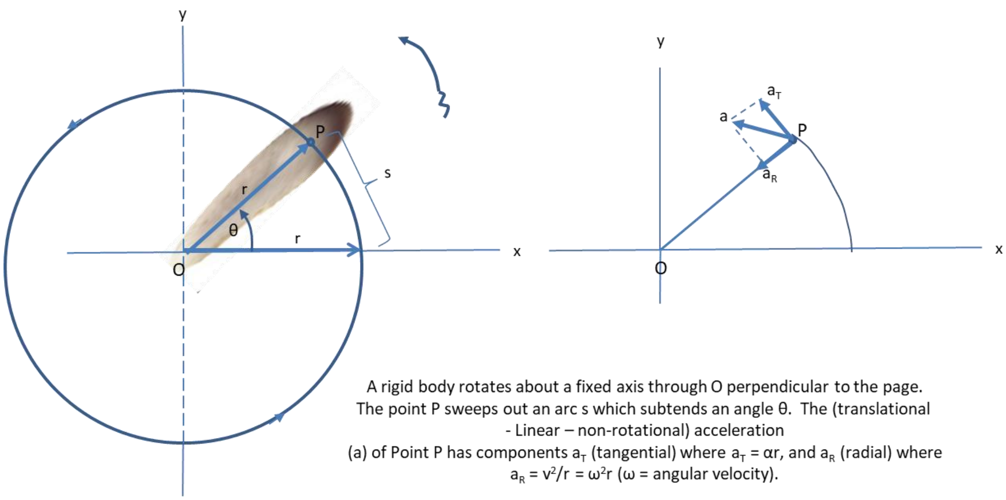


In the above illustrations, we represent angular velocity ω of the rotating rigid body by an arrow drawn along the axis of rotation as well as the rotation of a particle (such as P) about a fixed axis in the same way. The length of the arrow is made proportional to the magnitude of the angular velocity along the axis. By convention, if the fingers of the right hand curl around the axis in the direction of rotation of the body, the extended thumb points along the direction of the angular velocity vector (rad/sec). For the rigid body, the angular velocity vector will be in the positive z-direction. In the illustration, ω will be perpendicular to the page pointing up out of the page, corresponding to the counterclockwise rotation. The angular velocity of the turntable of a phonograph is a vector pointing down (since it turns clockwise). ***Nothing moves in the direction of the angular velocity vector! The vector represents the angular velocity of the rotational motion taking place in the plane perpendicular to it.***

Considering angular acceleration – is a vector. Since $\alpha = d\omega/dt$ in which $d\omega$ is a vector and dt a scalar, then α is a vector quantity. (Later it will be shown than torque, τ, and angular momentum are vector quantities).

Linear velocity, v and acceleration, a, apply to a particle moving in a circle. When a rigid body rotates about a fixed axis, every particle in the body moves in a circle. Hence we can describe the motion of such a particle either in linear variables or in angular variables. The relation between the linear and angular variables enables us to pass back and forth from one description to another and is very useful.

Consider a particle P in the rigid body, a distance r from the axis through O. This particle moves in a circle of radius r as the body rotates as illustrated below.



A rigid body rotates about a fixed axis through O perpendicular to the page. The point P sweeps out an arc s which subtends an angle θ. The (translational - Linear – non-rotational) acceleration (a) of Point P has components a_T (tangential) where $a_T = \alpha r$, and a_R (radial) where $a_R = v^2/r = \omega^2 r$ (ω = angular velocity).

Δv

The reference position is Ox. The particle moves through a distance s along the arc when the body rotates through an angle θ, such that s = θr (recall θ in radians is equal to s/r).

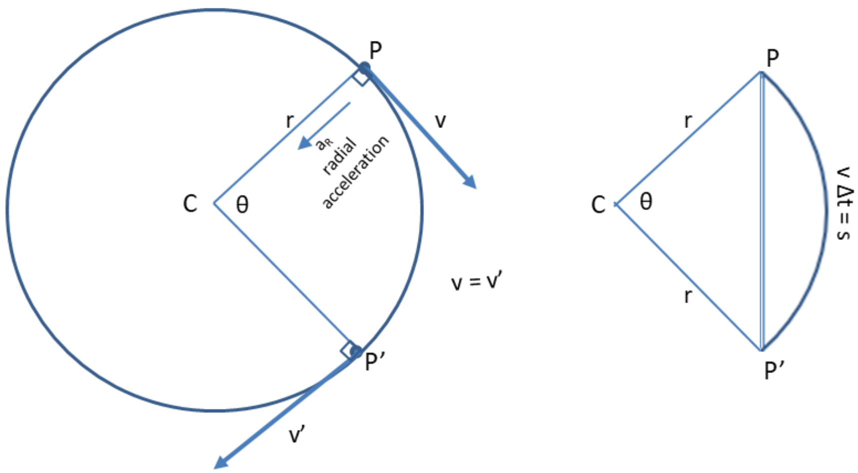
Differentiating both sides of this equation with respect to time and r is constant, we obtain… ds/dt = (dθ/dt)r, but ds/dt is the linear (tangential) speed (v) of the particle at P and dθ/dt is the angular (radial) speed ω of the rotating body so that v = ωr.

This is a relation between the magnitudes of the linear (tangential) velocity (v) and the angular (radial) velocity; the linear (tangential) speed (v) of a particle in circular motion is the product of the angular (radial) speed (ω) and the distance (r) of the particle from the axis of rotation.

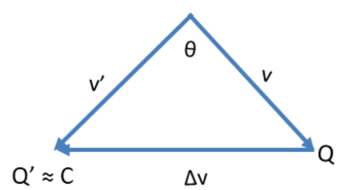
Differentiating this equation with respect to time (r is constant), we have dv/dt = r(dω/dt). But dv/dt is the magnitude of the tangential component (translational – linear – non-rotational) of acceleration (a = a_T) of the particle and dω/dt is the magnitude of the angular (radial) component of acceleration (α = a_R) of the rotating body, so that $a_T = \alpha r$.

Hence the magnitude of the tangential component of the linear acceleration (a_T) of a particle in circular motion is the product of the magnitude of the angular acceleration (α) and the distance r from the axis of rotation.

For a rigid body/particle moving in uniform circular motion (moving in a circle at constant speed, no acceleration), the tangential velocity vector changes continuously in direction but not in magnitude. Now to determine (radial) acceleration in uniform circular motion. Referring to the below Figure, let P be the position of the particle at time t and P' its position at the time t + Δt. The (tangential) velocity at P is v, a vector tangent to the curve at P. The velocity at P' is v', a vector tangent to the curve at P'. Vectors v and v’ are equal in magnitude, the speed being constant, but their directions are different. The length of the arc path, s, traversed during Δt (from P to P’) is the arc length PP’, which is equal to vΔt (since v = s/Δt), v being the constant speed.



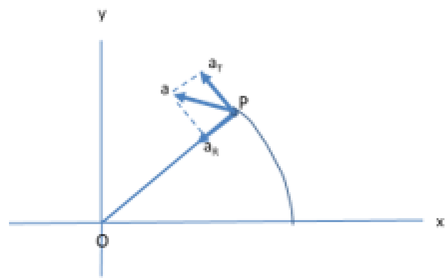
The vector diagram to the far right above has the same direction and magnitude as the vectors in the first illustration. This diagram allows us to visualize the change in vector velocity as the particle moved from P to P’. This change, v – v’ = Δv, is the vector which must be added to v to get v’. Notice that the Δv vector points inward, approximately toward the center of the circle.



Now the triangle OQQ’ formed by v, v’ and Δv is similar to the triangle CPP’ formed by chord PP’ and the radii CP and CP’. This is so because both the isosceles triangles having the same vertex angle; the angle θ between v and v’ is the same as the angle PCP’ (and QOQ’) because v is perpendicular to CP and v’ is perpendicular to CP’. We can therefore write (proportional rates): Δv/v ≈ s/r ≈ v Δt/r (approximately), the chord PP’ →equal to the arc length PP’ (for θ sufficiently small). This relation becomes more nearly exact as Δt is diminished, since the chord and the arc then approach each other in length. Notice that Δv approaches closer and closer to a direction perpendicular to v and v’ as Δt is diminished and therefore approaches closer and closer to a direction pointing to the exact center of the circle. It follows (since Δv/v ≈ v Δt/r) from this relation that $\Delta v/\Delta t \approx v^2/r$, the rate of velocity of change (= a_R, the radial acceleration), approximately, and in the limit when Δt → 0 this expression becomes exact. We therefore obtain the rate of velocity change:

$\lim_{\Delta t \to 0} \Delta v/\Delta t = dv/dt = v^2/r = a_R$ (radial acceleration) .

We can also express $a_R = v^2/r$ = α (angular acceleration) = $\omega^2 r$ (since v = ωr). The resultant linear acceleration (a) of point P (comprised of the components of radial a_R and tangential a_T) is shown in the illustration.



Thus the equations:

$$s = \theta/r \text{ (radian equation)}$$

$$v = \omega r \text{ (relating tangential velocity to radial speed)}$$

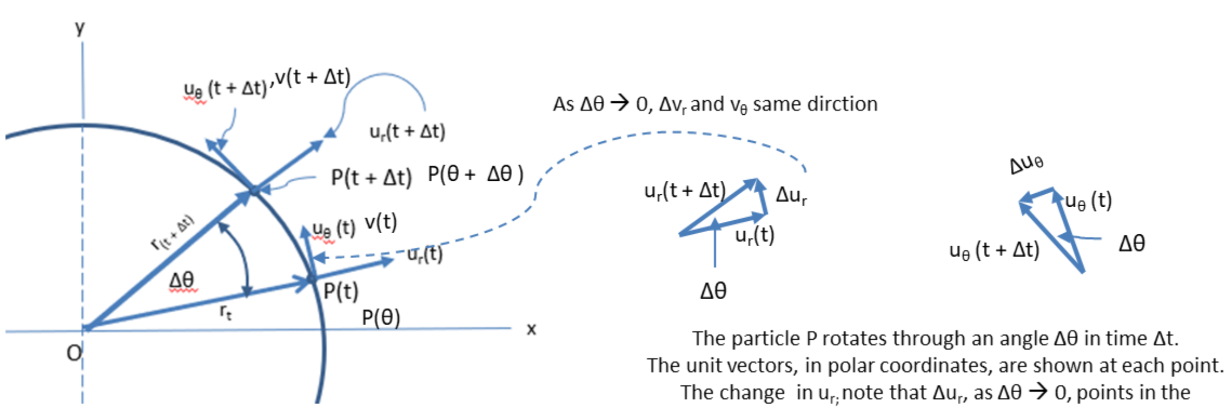
$$a_T = \alpha r \text{ (relating tangential acceleration to radial centripetal acceleration)}$$

$$\alpha = a_R = v^2/r = \omega^2 r \text{ (relating radial/centripetal acceleration to tangential velocity v or angular speed } \omega\text{)}$$

now enable us to describe the motion of one point on a rigid body rotating about a fixed axis either in angular variables or in linear variables. **The reason for the dual linear and angular systems is that different points on the same rotating rigid body do not have the same linear displacement (x), speed (v) or acceleration (a), but all points on a rigid body rotating about a fixed axis do have the same angular displacement (θ) , speed (ω), or acceleration (α) at any instant. By the use of angular variables we can describe the motion of the body as a whole in a simpler way.**

The prior analysis are relations between scalar quantities (magnitude only and not direction), both the linear and angular variables being expressed in scalar form. **Now to express in vector (magnitude and direction) form.**

The below Figure shows a particle P, rotating about a fixed axis (z) through the origin, at times t and t + Δt.



The particle P rotates through an angle Δθ in time Δt.
The unit vectors, in polar coordinates, are shown at each point.
The change in u_r; note that Δu_r, as Δθ → 0, points in the direction of u_θ. The change in u_θ; note that Δu_θ, as Δθ → 0, points in the direction of $-u_r$.

The particle moves in a circle of constant radius r; beyond this there are no restrictions on its motion and in general v (linear speed, (a) linear acceleration, ω (angular speed) and α (angular – radial – acceleration) may have values that vary as the particle moves. We express the restriction to a constant radius by

$\mathbf{r} = \mathbf{u_r} r$,

in which $\mathbf{u_r}$ is a **unit vector**[2] (magnitude of 1) in the direction of **r**.

Differentiating with respect to time (recall r but not **r** and $\mathbf{u_r}$ since their directions can change, is a constant),

$d\mathbf{r}/dt = d\mathbf{u_r}/dt\ (r)$

$d\mathbf{r}/dt$ (rate of change in the direction of r radius) is the linear vector velocity, **v**, of the particle. To evaluate $d\mathbf{u_r}/dt$, consider the above illustration that shows the unit vector $\mathbf{u_r}$ for two different positions of P [P(t), P(t + Δt)]; u_r (t), u_r(t + Δt)] , corresponding to a rotation through a (small) angle Δθ. Using the definition of angular measure in radians (θ in radians = s(arc)/r (radius); = s/r) we obtain the magnitude of the (vector) change $\Delta\mathbf{u_r}$ in $\mathbf{u_r}$, from

$\Delta u_\mathbf{r} = [u_\theta = (1)]\ \Delta\theta$; $[\theta = s/r;\ \Delta\theta \approx \Delta u_r/u_r,\ \Delta\theta\ u_r \approx \Delta u_r;\ u_r = 1;\ \Delta u_\mathbf{r} \approx (1)\ \Delta\theta]$,

in which the factor (1) [u_θ], reminds us that the two unit vectors have unit length. The above equation will be correct if Δθ is small enough so that we can neglect the difference in length between the chord ($\mathbf{u_r}$) and the arc (s) in the small triangle. The change in $\mathbf{u_r}$ (direction) is a vector, $\Delta\mathbf{u_r}$, whose magnitude is given by the above equation; its direction, assuming Δθ is small enough, is given by the unit vector $\mathbf{u_\theta}$ (vector of θ change). This follows because, if $\mathbf{\Delta u_r}$ is translated to point P, we see that, as $\Delta\theta \rightarrow 0$, it points in the direction of $\mathbf{u_\theta}$. Thus we find

$\Delta\theta = \Delta s/r$; $\Delta\theta \approx \mathbf{\Delta u_r / u_\theta}$; $\mathbf{\Delta u_r \approx u_\theta}\ \Delta\theta$; dividing by Δt, $\mathbf{\Delta u_r}/\Delta t \approx \mathbf{u_\theta}\ \Delta\theta/\Delta t$

And the limit: $\lim \Delta t \rightarrow$ $\mathbf{du_r}/dt = \mathbf{u_\theta}\ \Delta\theta/\Delta t = \mathbf{u_\theta}\ \omega$,

And since, $d\mathbf{r}/dt = d\mathbf{u_r}/dt\ (r)$; $\mathbf{v = u_\theta}\ \omega r$ (vector relationship); where the scalar relationship corresponds to v = ωr, and is one of the relationships obtained before connecting the linear speed v of a particle in circular motion with its angular speed ω.

To find the relation between linear and angular (vector) acceleration we differentiate $\mathbf{v = u_\theta}\ \omega r$ remembering that r is constant although $\mathbf{u_\theta}$ and ω can vary. We have

$d\mathbf{v}/dt = r[\mathbf{u_\theta}\ d\omega/dt + \omega\ \mathbf{du_\theta}/dt]$

Since dv/dt is the linear acceleration **a** and dω/dt is angular acceleration (radial, centripetal) **α**, we have…

$\mathbf{a} = r[\mathbf{u_\theta}\ \alpha + \omega\ \mathbf{du_\theta}/dt]$

Now considering r = $\mathbf{u_\theta} r$,

in which $\mathbf{u_\theta}$ is a unit vector (magnitude of 1) in the direction of **r**.

---

[2] See Addendum 1 for refresher of scalar, vector and unit vector.

Differentiating with respect to time (recall r is a constant but not **r** and $\mathbf{u}_\theta$ since their directions change),

d**r**/dt = d$\mathbf{u}_\theta$/dt (r)

d**r**/dt is the linear vector velocity, **v**, of the particle. To evaluate d$\mathbf{u}_\theta$/dt, consider the above illustration that shows the unit vector $\mathbf{u}_\theta$ for two different positions of P, corresponding to a rotation through a (small) angle Δθ. Using the definition of angular measure in radians (θ in radians = s(arc)/r (radius); = s/r) we obtain the magnitude of the (vector) change Δ$\mathbf{u}_\theta$ in $\mathbf{u}_\theta$, from

$\Delta u_\theta = (1)\ \Delta\theta$; [$\theta = s/r$; $\Delta\theta \approx \Delta u_\theta/u_\theta$, $\Delta\theta\ u_\theta \approx \Delta u_\theta$; $u_\theta = 1$; $\Delta u_\theta \approx (1)\ \Delta\theta$],

in which the factor (1) reminds us that the two unit vectors have unit length. The above equation will be correct if Δθ is small enough so that we can neglect the difference in length between the chord (Δ$\mathbf{u}_\theta$) and the arc (s) in the small triangle. The change in $\mathbf{u}_\theta$ is a vector, Δ$\mathbf{u}_\theta$, whose magnitude is given by the above equation; its direction, assuming Δθ is small enough, is given by the unit vector $\mathbf{u}_r$ **(vector of radius direction change)**. This follows because, if **Δ**$\mathbf{u}_\theta$ is translated to point P, we see that, as Δθ →0, it points in the direction of -$\mathbf{u}_r$ ($\mathbf{u}_\theta$ vector points toward or against $\mathbf{u}_r$ and not in the same direction). Thus we find

$\Delta\theta = \Delta s/r$; $\Delta\theta \approx \Delta\mathbf{u}_\theta / -\mathbf{u}_r$; $\Delta\mathbf{u}_\theta \approx -\mathbf{u}_r\,\Delta\theta$; dividing by Δt, $\Delta\mathbf{u}_\theta/\Delta t \approx -\mathbf{u}_r\,\Delta\theta/\Delta t$

And the limit: limΔt→ $d\mathbf{u}_\theta/dt = -\mathbf{u}_r\,\Delta\theta/\Delta t = -\mathbf{u}_r\,\omega$.

The minus sign comes in because when we translate **Δ**$\mathbf{u}_\theta$ to point P, we see that as Δθ→0, it points radially inward (toward the center of the circle of motion), in the direction opposite to $\mathbf{u}_r$, so from a vector perspective a negative direction.

Making the substititions, we now have…

(a) = r[$\mathbf{u}_\theta$ α + ω d$\mathbf{u}_\theta$/dt] and d$\mathbf{u}_\theta$/dt = -$\mathbf{u}_r$ ω; **a** = r[$\mathbf{u}_\theta\,\alpha - \omega^2\mathbf{u}_r$] or (a) = $a_r + a_t$;,(where $\mathbf{u}_\theta$ and $\mathbf{u}_r$ are unit (1) vectors).

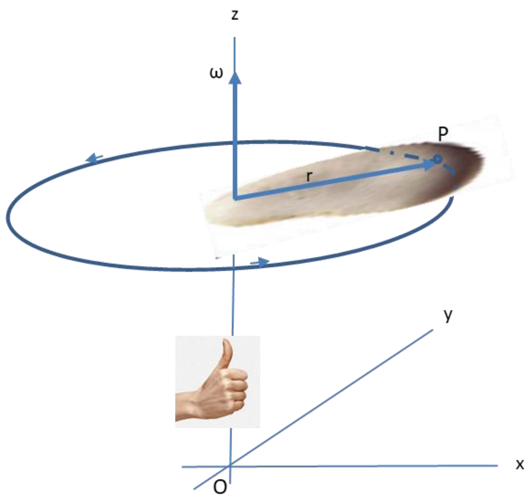
We know that **a** has a radial (or centripetal) component $\mathbf{a}_R$ and a tangential component $\mathbf{a}_T$. Their magnitudes from the above equation are:

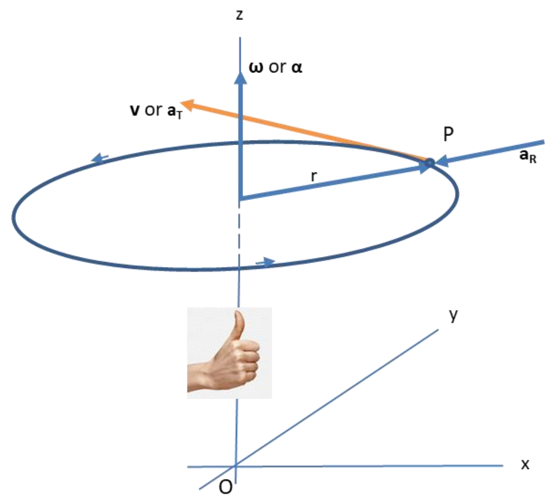
$\mathbf{a}_R = r\alpha\ (1)$; and

$\mathbf{a}_T = r\omega^2\ (1) = v^2/r$ (where $v = \omega r$, $v^2 = \omega^2 r^2$)

Equations $\mathbf{v} = \mathbf{u}_\theta\omega r$ and $\mathbf{a} = r[\mathbf{u}_\theta\,\alpha - \omega^2\mathbf{u}_r]$ are relations between the linear kinematic variables in vector form and the angular kinematic variables in scalar form. Now to derive relationship for both to be in vector form.

The below Figure shows the vectors **r, v,** $\mathbf{a}_T$**,** $\mathbf{a}_R$**, ω** and **α** for the rotating particle P.





The angular quantities are on the axis of rotation (z), pointing in the direction given by the right-hand rule. We declare, and shall prove, that the relationships we seek are:

v = ω x r, and

a = a_T + a_R

in which **a_T = α x r and a_R = ω x v…**

Vector analysis indicates that if **c = a x b**, then the magnitude of **c** is |a| |b| sin θ, where θ is the angle between vectors **a** and **b**. In the above Figure, note that **ω** and **r, ω** and **v**, and **α** and **r** are each mutually perpendicular to each other; thus the angle θ for each of these three pair of vectors is 90°.

VECTOR MULTIPLICATION

Vectors when added together must be of like kind (length plus length, but not length plus temperature). In contrast, different kinds of vectors can be multiplied by each other and provide a new vector ((i) 2 vectors to yield a scalar, or (ii) multiply a vector and a scalar and (iii) multiply two vectors to yield another vector). Unit vectors are typically cited as $i^{\wedge}_{x}$ "I" hat for the x cartisian plane, $j^{\wedge}_{y}$ and $k^{\wedge}_{z}$. The new vector has the same direction as **a** if $k^{\wedge}_{z}$ (z coordinate unit vector, since the vector is perpendicular to the xy plane)) is positive (+) and opposite if $k^{\wedge}_{z}$ is negative (-). To divide a vector by a scalar, multiply the vector by the reciprocal of the scalar.

When multiplying a vector quantity by another vector quantity, we must distinguish between the scalar (or dot) product and the vector (or cross) product.

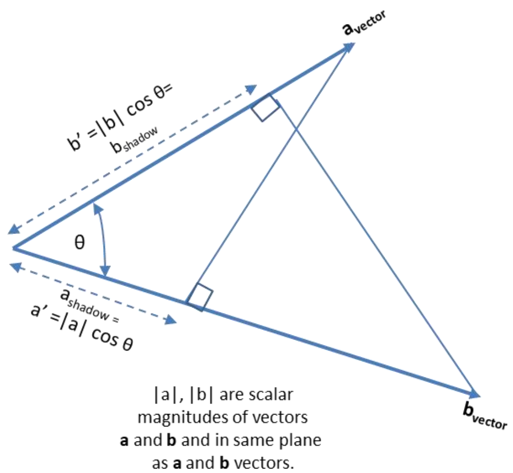
DOT PRODUCT

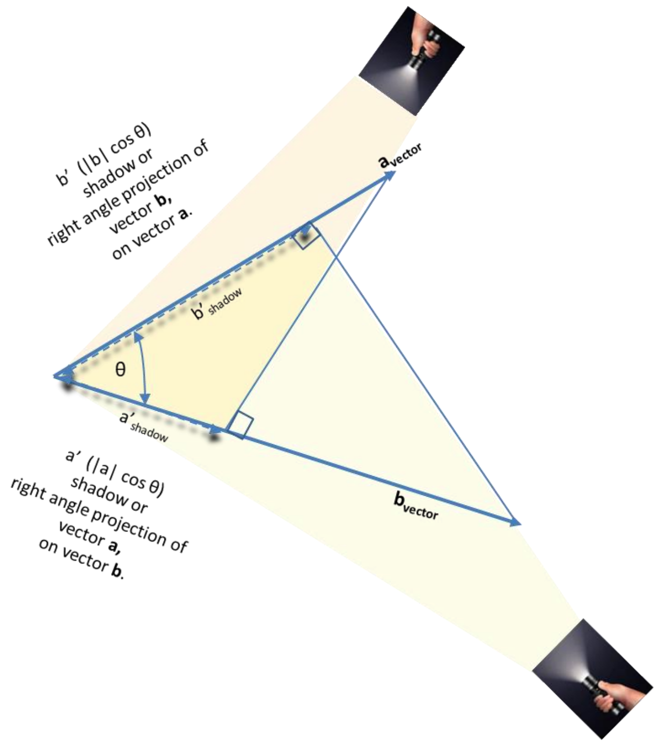
The scalar or **dot product** of two vectors **a** and **b**, written as **a · b (a dot b)**, is defined as

a · b = |a|$_{| |\text{=magnitude of vector a}}$|b| cos θ, (a scalar result, has magnitude but direction is not relevant) [**a,b** are vectors and |a| and |b| their magnitudes]

where |a| is the magnitude of vector **a**, |b| is the magnitude of vector **b** and cos θ is the cosine of the angle between the two vectors (always take the smaller of the angle since there are always two angles associated with two vectors, a small and large one, θ and 360° – θ).

In math, "associative" describes a property where the way numbers are grouped with parentheses does not change the final result of an addition or multiplication operation. This means you can change the grouping of numbers without affecting the sum or product. The associative property applies to addition and multiplication but not to subtraction or division.



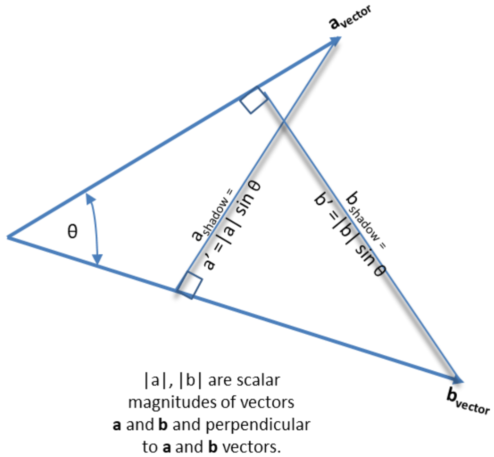


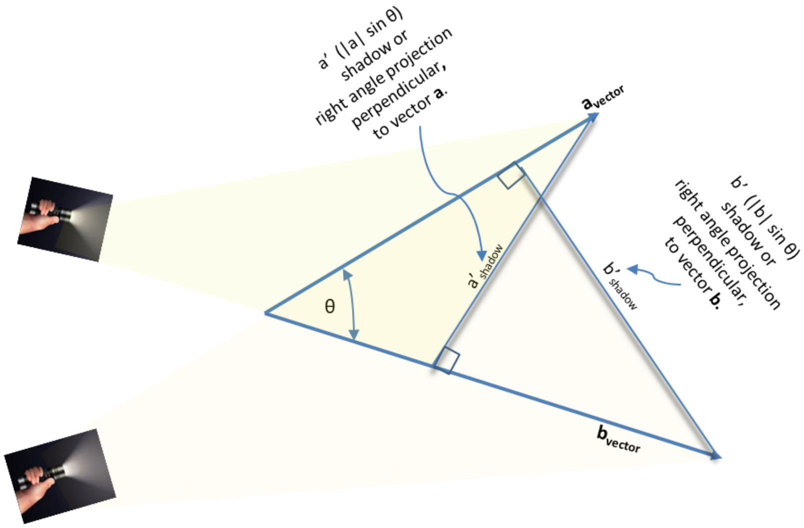
Since |a| and |b| are scalars and cos θ is a pure number, the scalar product of two vectors is a scalar. In otherwords, |a|, |b| and cos θ are math numbers that can be multiplied to each other in any order and the result is the same. [Ex. 1 times 2 time 3 = 6; 3 times 2 times 1 = 6 or |a| |b| cos θ same as |a cos θ |b|]]. The scalar product of two vectors can be regarded as the product of the magnitude of one vector (|a| or |b|) and the component of the other vector (|a| cos θ or |b| cos θ) in the direction of the first.

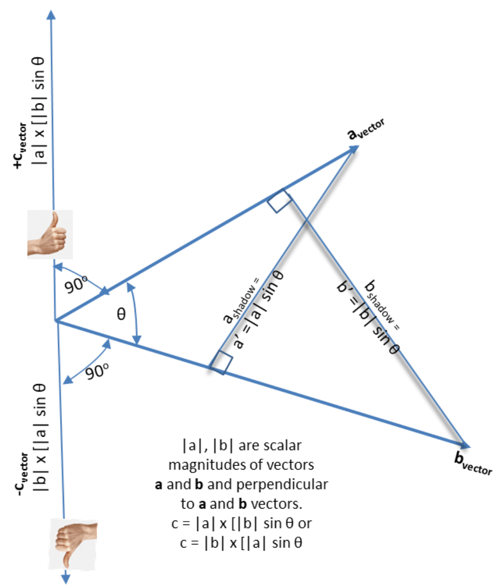
CROSS PRODUCT

The vector product of two vectors **a** and **b** is written **a x b** (cross product) and is another vector **c**, where **c = a x b**. The magnitude of **c** is defined as

c = |a| |b| sin θ (a vector result where **c** has magnitude and direction and |a| and |b| are the magnitudes), where θ is the angle between **a** and **b**.







The direction of **c**, the vector cross product of **a** and **b**, is defined to be perpendicular to the plane formed by **a** and **b**, and determined by the right hand thumb rule. Note that **b x a** is not the same as **a x b**, so order of factors in a vector cross product is important (order matters, not associative, a x b ≠ b x a) – magnitude the same but different direction (the right hand thumb rule will be pointed in opposite directions depending on which magnitude is chosen first).

2. TORQUE, ANGULAR MOMENTUM AND MOMENT OF INERTIA

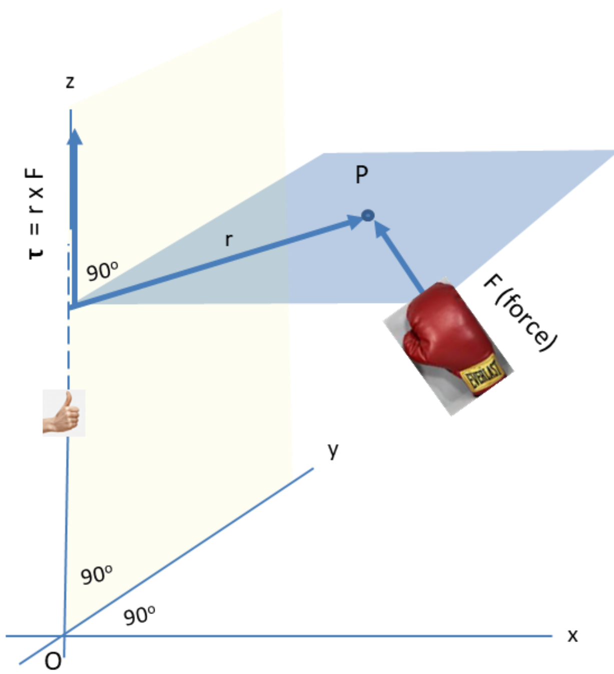
Section 2. considers the causes of rotation or rotational dynamics, and will assess both fixed and not fixed axis of rotation regarding **angular momentum** and **moment of inertia**.

In translational (linear) motion we associate a force F (Newtonian, = ma, mass times acceleration) with the linear acceleration (a) of a body with mass m. In rotational motion, what quantity shall we associate with the angular acceleration α of a body[3]?

TORQUE

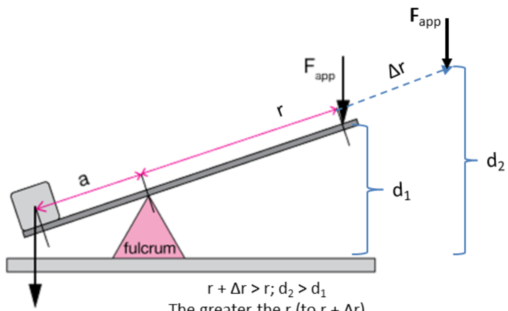
We shall call the **rotational analogue of linear (F = ma) force** – ***torque*** – **τ,** and shall define it for now as the special case of a single particle observed from an inertial reference frame. Later we shall extend the torque concept to systems of many particles (including rigid bodies) and shall show that torque is intimately associate with angular acceleration.

If a force **F** (vector, having magnitude and direction) acts on a single particle at point P whose position with respect to the origin O of a selected inertial reference frame is given by the displacement vector **r** (the leverage distance or moment arm), the torque **τ,** acting on the particle with respect to the origin O is defined as **τ = r x F** (this is like a 'leveraged' analysis where r (the leverage arm distance) provides the 'leverage', greater the r, or greater the leverage arm, like lifting a rock with a stick placed under a fulcrum, then greater the leverage and greater the torque).



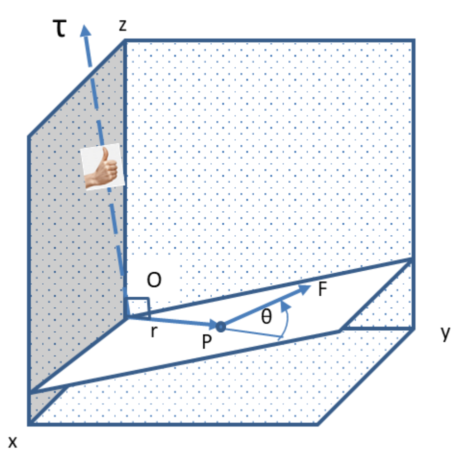
The direction of the vector **τ** of a particle, perpendicular to the r moment arm and Force plane.

[3] It cannot simply be force (F) because, as when dealing with a heavy revolving door, a given Force (vector) can produce various angular accelerations of the door depending on where the force is applied and how it is directed; a force applied to the hinge line cannot produce any angular acceleration, whereas a force of given magnitude applied at right angles to the door at its outer edge produces a (maximum) acceleration.



Torque is a vector quantity (has magnitude and direction). Its magnitude is given by

τ = **r** x F = |r|(|F| sin θ), a cross product, **r** and **F** order matters. Torque vector is perpendicular to the rF geometric plane, where θ is the angle between **r** and **F**; and its direction is normal to the plane formed by **r** and **F**. The sense (torque 'direction') is given by the right-hand rule for the vector product of two vectors, namely, one swings r into F through the smaller angle between them with the curled fingers of the right hand; the direction of the extended thumb then gives the direction of **τ.**



A force **F** is applied to a particle P (cross product of **r** x **F**, **F** = |F|sin θ), displaced by **r** relative to the origin O. The force vector make an angle θ with the radius vector **r.** The torque **τ** vector about O is shown. Its direction is perpendicular to the plane Formed by **r** and **F** with the sense given by the right-hand rule.

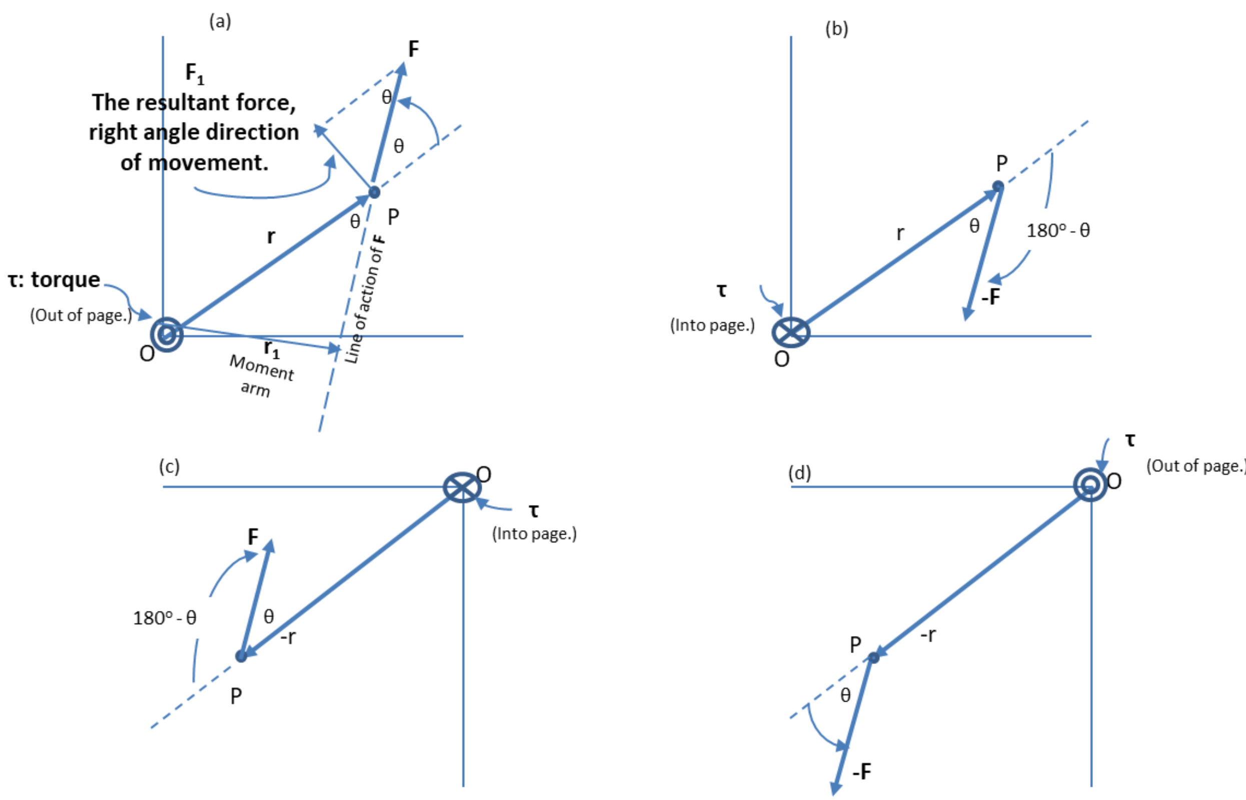
Torque has the same dimensions as force times distance, or in terms of our assumed fundamental dimensions, M (mass), L (length), T (time), it has the dimensions of ML^2T^{-2} (such as Newton-Meter – $(kg.m/s^2)m$; Foot-$Pounds_{force}$ – typical torque wrench dimensions used to torque lug nuts on a car wheel;). These are the same dimensions as work (force times distance). However, torque and work are very different physical quantities[4]. Torque is a vector (has magnitude and direction)

[4] Power is Work per unit of time or W/t = F x d/t = F x v. Torque wins races; horsepower sales cars. You can have a lot of horsepower but not much torque which is the desired result to turn the racing car axle quickly.

and work is a scalar (has magnitude but not direction), for example. The unit of torque may be the nt-meter or lb-ft, among other possibilities.

Note in the above illustration that the torque produced by a force depends not only on the magnitude and on the direction of the force but also on the point of application of the force relative to the origin, that is, location on the vector **r.** In particular, when particle P is at the origin O, so that the line of action of **F** passes through the origin, **r** is zero and the torque τ about the origin is zero. The farther out on **r** the bigger the torque (or as in the fulcrum example, greater leverage is applied though a longer distance to move an object).

Referring to the below Figures for torque illustrations…



The plane shown is that defined by **r** and **F**.

(a) The magnitude of **τ** (torque) is given by ($r_{1\text{-moment arm}}$ **F**; or $Fr_{\underline{1}}$ or ($F_{1\text{ force component}}$)r or (F sinθ)r or $rF_{\underline{1}}$, where $F_{\underline{1}}$ = F sinθ
(b) Reversing **F** reverses the direction of **τ**.
(c) Reversing **r** reverses the direction of **τ.**
(d) Reversing **F** and **r** leaves the direction of **τ** unchanged.

we can write the magnitude of τ as

$\tau = r\,(F\sin\theta) = rF_{\underline{1}}$

Torque is often called the **moment of force** and r is called the **moment arm**. As shown in the Figure, only the component of **F** perpendicular to **r** contributes to the torque. In particular, when θ is 0 or 180°, there is no perpendicular component (since sin of 0 or 180° = 0), then the line of action of the force passes through the origin and the moment arm r about the origin is zero.

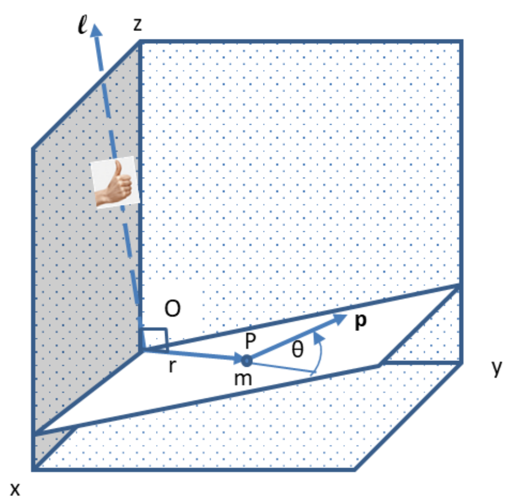
ANGULAR MOMENTUM OF A PARTICLE

We have found linear momentum to be useful in dealing with the translational motion of single particles or of systems of many particles (including rigid bodies). For example, linear momentum is conserved in collisions. For a single particle the linear momentum is **p (a vector)** = m**v (where velocity is a vector);** for a system of mupltiple particles it is **P** = M$\mathbf{v}_{cm}$ in which M is he total system mass and $\mathbf{v}_{cm}$ is the velocity of the center of mass.

In rotational motion, what is the analog of linear momentum?

Rotational momentum is called angular momentum and is defined below for the special case of a single particle. Later we shall broaden the definition to include systems of many particles and show that angular momentum, as we define it, is as useful a concept in rotational motion as linear momentum is in translational motion.

Consider a particle of mass m and linear momentum **p** (m**v**) at a position **r** relative a the origin O of an inertial reference frame (see below Figure).



A particle of mass m is at point P displaced **r** relative to the origin O, and has linear momentum **p** (m**v).**
The vector **p** makes an angle θ with the radius vector **r**.
The angular momentum ***ℓ*** of the particle with respect to origin O is shown.
Its direction is perpendicular to the plane formed by **r** and **p** with the sense given by the right-hand rule.

We define the angular momentum ***ℓ*** of the particle with respect to the origin O to be

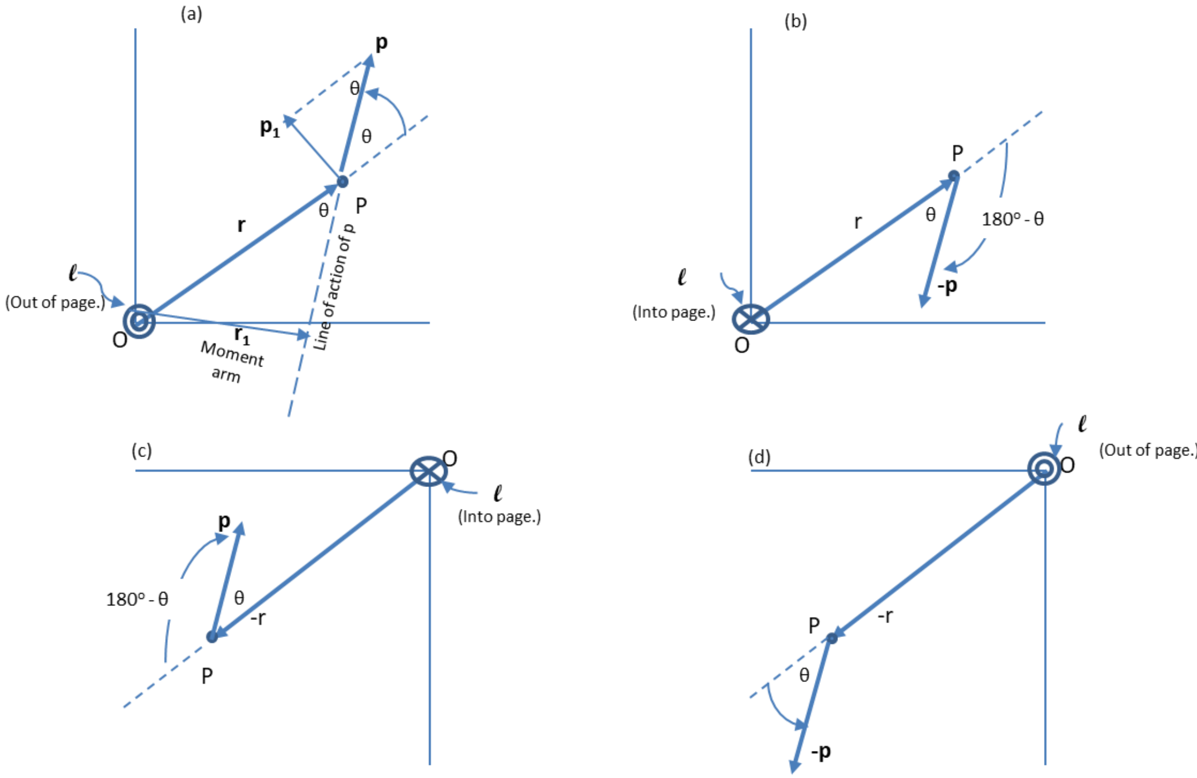
***ℓ* = r x p** (a cross product, order matters)

Angular momentum is a vector. Its magnitude, similarly determined as above for torque is given by

$\ell = |r|\ [|p|\ \sin\theta]$

where θ is the angle between **r** and **p**; its direction like torque is normal to the plane formed by **r** and **p**. The sense is given by the right-hand rule, namely, one swings **r** into **p**, through the smaller angle between them, with the curled fingers of the right hand; the extended right thumb then point in the direction of angular momentum ***ℓ***.

Similar as torque,



The plane shown is that defined by **r** and **p**.

(a) The magnitude of ***ℓ*** is given by ($r_{moment\ arm}$ **p**or [|p| sinθ] |r| or rp_1, where p_1 = |p| sinθ

(b) Reversing **p** revereses the direction of ***ℓ***.

(c) Reversing **r** reverses the direction of ***ℓ***.

(d) Reversing **p** and **r** leaves the direction of ***ℓ*** unchanged.

we can write the magnitude of angular momentum ***ℓ*** either as

$\ell = [\,|p| \sin\theta]r = rp_1$,

in which $p_1 = |p| \sin\theta$ is the component of **p** at right angles to **r**.

Angular momentum is often called the moment of (linear) momentum and r is often called the moment arm. Only the component of **p** perpendicular to **r** contributes to the angular momentum. When the angle θ between **r** and **p** is 0 or 180°, there is no perpendicular component (since sin of 0 or 180° = 0), then the line of action of momentum passes through the origin and the moment arm r about the origin is zero.

We now derive an important relation between torque and angular momentum. We have seen that **F** = d(m**v**)/dt = d**p**/dt for a particle. Let us take the vector cross product of **r** with both sides of this equation, obtaining

r x **F** = **r** x d**p**/dt

But **r** x **F** is torque, or moment of a force, about O. We can then write

τ = **r** x d**p**/dt

Differentiating the definition of angular momentum, $\boldsymbol{\ell}$ = **r x p**, yields

$d\ell/dt = d(\mathbf{r} \times \mathbf{p})/dt$.

The derivative of a vector product is taken in the same way as the derivative of an ordinary product, except that we ***must not change the order of the terms***. We have…

$d\ell/dt = d\mathbf{r}/dt \times \mathbf{p} + r \times d\mathbf{p}/dt$.

But d**r** is the vector displacement (Δr_1) of the particle in the time dt so that d**r/**dt is the instantaneous (tangential) velocity **v** of the particle. Also, **p** = m**v**, so that the equation can be rewritten as

$d\ell/dt = \mathbf{v} \times m\mathbf{v} + r \times d\mathbf{p}/dt$.

Now **v x** m**v** = 0, because the **vector cross** product of two parallel vectors is zero. [θ between v and sin v is 0; therefore sin θ or sin (0^o) = 0; v x mv sin θ = 0]

Therefore, $d\ell/dt = r \times d\mathbf{p}/dt$ and since $\tau = \mathbf{r} \times d\mathbf{p}/dt$, we have

$\tau = d\ell/dt$

which states that the time rate of change of the angular momentum ($d\ell/dt$) of a particle is equal to the torque acting on it. This result is the rotational analog of linear momentum (were **F** = m**a** = md**v**/dt = d(m**v**)/dt = d**p**/dt), which stated that the time rate of change of the linear momentum of a particle is equal to the force acting on it, that is, that **F** = d**p**/dt.

The equation, $\tau = d\ell/dt$, like all vector equations, is equivalent to three scalar equations, namely

$\tau_x = (d\ell 1/dt)_x \quad \tau_y = (d\ell/dt)_y \quad \tau_z = (d\ell/dt)_z$

Hence, the x-component of the applied torque is given by the x-component of the change with time of angular momentum. Similar results hold for y and z directions.

NOW FOR SYSTEMS OF PARTICLES…

Now consider a system of many particles. To calculate the total angular momentum $\mathcal{L}$ of a system of particles about a given point, we must add vectorially the angular momenta of all the individual particles of the system about this same point. For a system containing n particles we have, then,

$$\mathcal{L} = \ell_1 + \ell_2 + \ldots + \ell_n = \sum_{i=1}^{n} (\ell_i)$$

in which the (vector) sum is taken over all particles in the system.

As time goes on, the total angular momentum $\mathcal{L}$ of the system about a fixed reference point (which we choose, as in our basic definition of $\mathcal{L}$ = **r** x **p**, to be the origin of an inertial reference frame) may change. This change, $d\mathcal{L}/dt$, can arise from two sources: (1) torques exerted on the particles of the system by internal forces between the particles and (2) torques exerted on the particles of the system by external forces.

Given that Newton's third law holds in its so called strong form, that is, if the forces between any two particles not only are equal and opposite but are also directed along the line joining the two particles, then the total internal torque is zero because the torque resulting from each internal action-reaction force pair is zero.

Hence the first internal forces source contributes nothing.

For our reference point, therefore, only the second external source torque remains, and we can write

$\tau_{external}$ = (**d** $\mathcal{L}$ **/**dt),

where $\tau_{external}$ stands for the sum of all the external torques acting on the system. In other words, the time rate of change of the total angular momentum of a system of particles about the origin of an inertial reference frame is equal to the sum of the external torques acting on the system.

$\tau_{external}$ = (**d** $\mathcal{L}$ **/**dt) is the generalization of τ = **dℓ/**dt to many particles. When we have one particle, there are no internal forces or torques. $\tau_{external}$ = (**d** $\mathcal{L}$ **/**dt) holds whether the particles that make up the system are in motion relative to each other or whether they have fixed spatial relationships, as in a rigid body.

$\tau_{external}$ = (**d** $\mathcal{L}$ **/**dt) is the rotational analog of $\mathbf{F}_{ext}$ = d**P**/dt

which tells us that for a system of particles (rigid body or not) the resultant external force acting on the system equals the time rate of change of the linear momentum of the system.

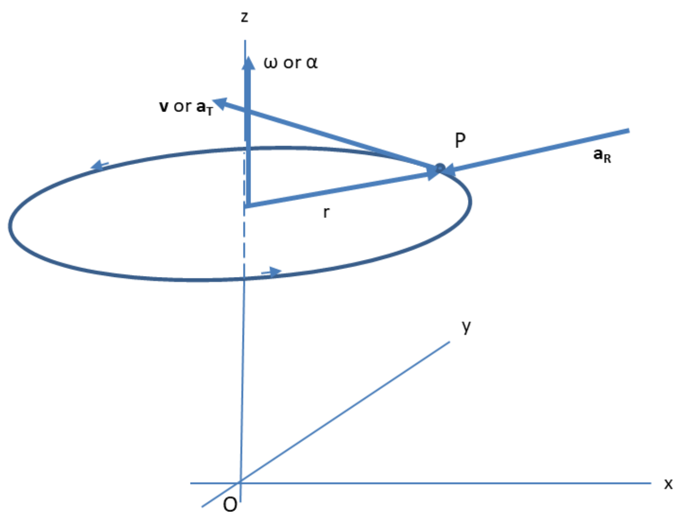
$\tau_{external}$ = (**d** $\mathcal{L}$ **/**dt) as defined, holds when **τ** and $\mathcal{L}$ are measured with respect to the origin of an inertial reference frame. We may well say whether it still holds if we measure these two vectors with respect to an arbitrary point (a particular particle, for example) in the moving system. In general, such a point would move in a complicated way as the body or system of particles translated, tumbled, and changed its confituration and $\tau_{external}$ = (**d** $\mathcal{L}$ **/**dt) would not apply to such a reference point. However, if the reference point is not fixed in our reference frame, then $\tau_{external}$ = (**d** $\mathcal{L}$ **/**dt) does hold. This is another remarkable property of center of mass. Thus we can separate the general motion of a system of particles into the translational motion of the center of mass, $\mathbf{F}_{ext}$ = d**P**/dt.

MOMENT OF INERTIA

We shall now turn our attention to an important special case of a system of many particles, a rigid body.

In a rigid body the particles in the system always maintain the same positions with respect to one another. In studying the rotation of a rigid body we shall consider first the special case, often encountered, in which the ***axis of rotation is fixed in an inertial frame of reference***. "**Fixed axis of rotation**" is in reference to the general motion of a system of many particles into translational motion of its center of mass and rotational motion about its center of mass. Hence the considerations discussed herein apply also to rotations about an axis that is not fixed in an inertial reference frame, provided (1) the axis passes through the center of mass and (2) the moving axis always has the same direction in space, that is, the axis at one instant is parallel to the axis at any other instant. Although reference is made to a "fixed [parallel] axis" in this discussion it always means to include this special case of a moving axis.

Imagine a rigid body rotating with angular speed ω about an axis that is fixed in a particular intertial reference frame (see below Figure).



The directions of the vectors **r, v, a_T, a_R, ω** and **α** for a particle rotating in a circle about the z-axis.

Each particle in such a rotating body has a certain amount of kinetic energy. A particle of mass m as a distance of r from the axis of rotation moves in a circle of radius r with an angular speed of ω about this axis and has a linear speed v = ωr. Its kinetic energy therefore is $1/2mv^2 = 1/2mr^2\omega^2$. The total kinetic energy of the body is the sum of the kinetic energies of its particles.

If the body is rigid, as we assume in this section, ω is constant for all particles. The radius r may be different for different particles. Hence the total kinetic energy K of the rotating body can be written as

$K = ½(m_1r_1^2 + m_2r_2^2 + \ldots\ m_nr_n^2)\omega^2 = \sum_{i=1}^{n} (m_ir_i^2)\omega^2$.

The term $\sum (m_ir_i^2)$ is the sum of the products of the masses of the particles by the squares of their respective distances from the axis of rotation. If we denote this quantity as I, then

$I = \sum (m_ir_i^2)$

is called the rotational inertia, or moment of inertia, of the body with respect to the particular axis of rotation. Note that the rotational inertia of a body depends on the particular axis about which it is rotating as well as on the shape of the body and the manner in which its mass is distributed. Rotational inertia has the dimensions of ML^2 and is usually expressed in kg-m^2 or slug-ft^2.

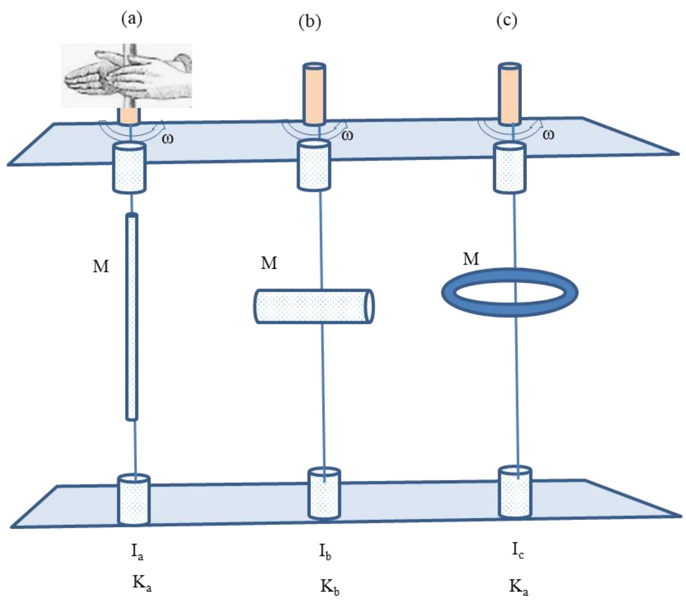
In terms of rotational inertia we can now write the kinetic energy of the rotating rigid body as

$K = ½\ I\ \omega^2$.

This is analogous to the expression for the kinetic energy of translation of a body, $K = 1/2Mv^2$. We have already seen that the angular speed ω is analogous to the linear speed v. Now we see that the rotational inertia I is analogous to the mass, or the translational inertia M. Although the mass of a body does not depend on its location, the rotational inertia of a body does depend on the axis about which it is rotating.

The rotational kinetic energy given by $K = ½\ I\ \omega^2$ is simply the sum of the ordinary translational kinetic energy of all the parts of the rigid body and not a new kind of energy. Rotational kinetic energy is simply a convenient way of expressing the kinetic energy for a rotating rigid body.

$I = \sum (m_i r_i^2)$ and $K = \frac{1}{2} I \omega^2$ show that the rotational energy of a body, for a given angular speed ω, depends not only on the mass of the body but also on the way that mass is distributed round the axis of rotation. The illustration in the below Figure makes this convincing.



An illustration to show that $I_a < I_b < I_c$. The three lead bodies have the same mass M but the mass is distributed differently about the axis of rotation.

This illustration shows three identical aluminum shafts, to each of which is attached a body of mass M, made of lead. In (a) the mass is very close to the shaft so that the quantities r_i in $I = \sum (m_i r_i^2)$ for the particles that make up the body are relatively small; in (b) the particles, on average, farther from the shaft and in (c), in which the body is a flywheel, they are still further, corresponding to still larger value of r_i.

Now let us twist each handle until each shaft, starting from rest, is spinning at the same measured anglar speed, ω. We know from experience that we shall need to do relatively little work (effort or external Force) on shaft (a), some what more work on shaft (bi), and still more on shaft (c). In fact, if we were not certain which body was attached to which shaft {because they were hidden behind a curtain} we could label the shafts with confidence using this technique. Since the work (force times distance) on each shaft is equal to the kinetic energy $1/2I\omega^2$ imparted to each shaft, the experimental result, that $K_a < K_b < K_c$ when each shaft has the same angular speed ω leads to the conclusion that $I_a < I_b < I_c$. This is just what we expect from the defining equation or $I = \sum (m_i r_i^2)$.

Just as the mass M, which we may call the translational inertia, is a measure of the resistance a body offers to a change in its translational motion, so I, the rotational inertia, is a measure of the resistance a body offers to a change in its rotational motion about a given axis.

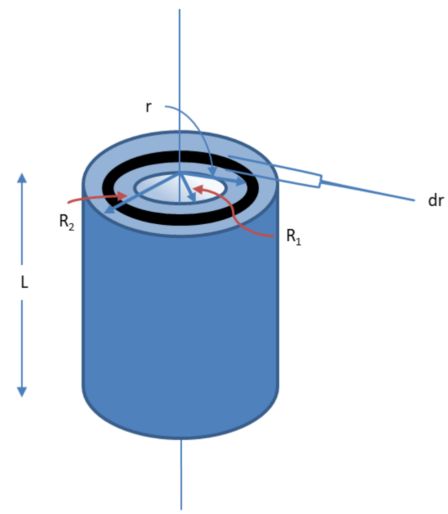
For a body that is not composed of discrete point masses but is instead a continuous distribution of matter, the summation of $I = \sum (m_i r_i^2)$ becomes an integration. We imagine the body to be subdivided into infinitesimal elements, each of mass dm. We let r be the distance from such an element to the axis of rotation. Then the rotational inertia is obtained from

$I = \int (r^2 dm)$

where the integral is taken over the whole body. The procedure by which the summation ∑ of a discrete distribution is replaced by the integral ∫ for a continuous distribution is the same as that discussed for the center of mass.

For bodies of irregular shape the integrals may be hard to evaluate. For bodies of simple geometrical shape the integrals are relatively easy when an axis of symmetry is chose as the axis of rotation.

Let us illustrate the procedure for an annular cylinder (or ring) about the cylinder axis using the below illustration.



Calculating the rotational inertia of an annular cylinder.

The most convenient mass element is an infinitesimally thin cylinder shell of radius r, thickness dr, and length L. If the density (ρ) of the material, the mass per unit volume, is C, then

$dm = \rho\, dV$, where dV is the volume of the cylindrical shell of mass dm. We have

$dV = (2\pi r\, dr)L$, so that $dm = 2\pi L\, \rho r\, dr$

Then the rotational inertia about the cylinder axis is

$I = \int r^2\, dm = 2\pi L \int_{R1}^{R2} \rho r^3 dr$

Here R_1 is the radius of the inner cylindrical wall and R_2 is the radius of the outer cylindrical wall.

If the body did not have a uniform constant density, we would have to know how ρ depends on r before we could carry out the integration. Let us assume for simplicity that the density is uniform. Then

$I = 2\pi L\rho \int_{R1}^{R2} r^3 dr = (2\pi L\rho/4)\,(R_2^4 - R_1^4) = \{\, \pi\rho(R_2^2 - R_1^2)\}_M\, L\, [(R_2^2 - R_1^2)]/2$

The mass M of the annular cylinder is the product of the density ρ by its volume $(\pi(R_2^2 - R_1^2)\, L$, or

$M = \rho\pi(R_2^2 - R_1^2)\, L$.

The rotational inertia of the annular cylinder (or ring) of mass M, inner radius of R_1 and outer radius of R_2, is therefore

$I = ½\, MR^2$

about the cylinder axis, where R is the radius of the solid cylinder of mass M.

A hoop can be thought of as a very thin-walled cylinder. In that case

$R_1 \approx R_2 \approx R$, and $I = MR^2$

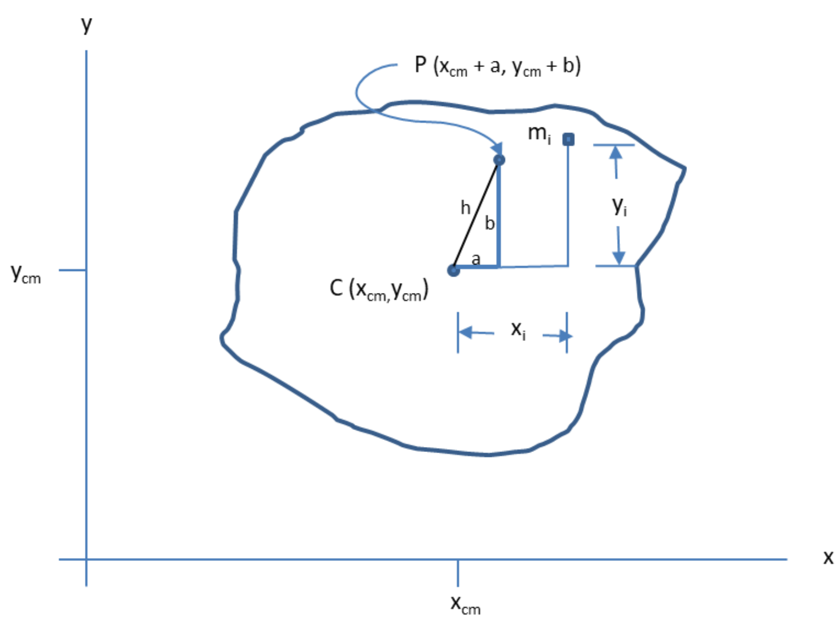
is the moment of a hoop of mass M and Radius R about the cylinder axis.

This result for the thin hoop is obvious since every mass point in the hoop is the same distance R from the central axis. For the solid cylinder (or disk) having the same mass as the hoop, the rotational inertia (or moment of inertia) would naturally be less than that of the hoop, because most of the cylinder (or disk) is less than a distance R from the axis.

There is a simple and very useful relation between the rotational inertia I of a body about any axis and its rotational inertia I_{cm} with respect to a parallel axis through the center of mass. If M is the total mass of the body and h the distance between the two axes, the relation is

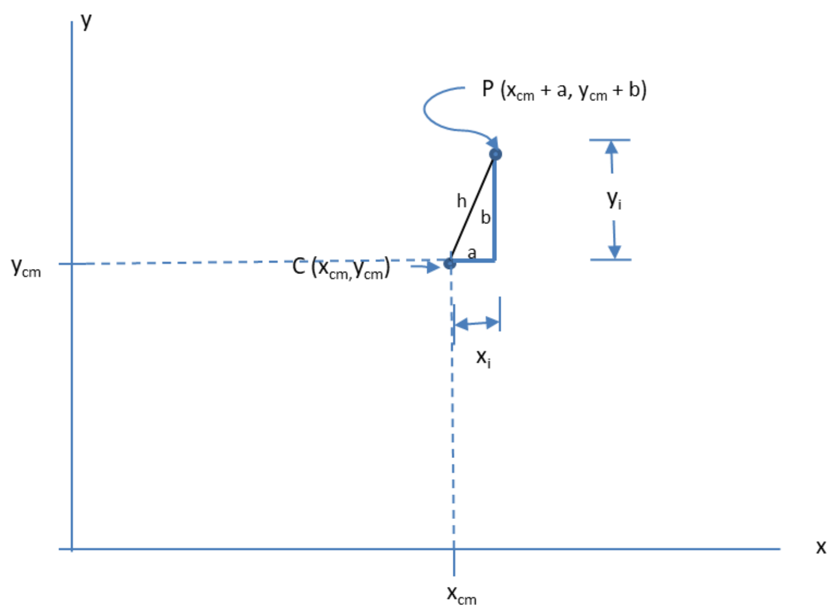
$I = I_{cm} + Mh^2$

Proof of this relationship follows making reference to the below Figure.



Derivative of the parallel-axis theorem. Knowing the rotational inertia about an axis through C, we can find its value about a parallel axis through P.

The proof of this relation (parallel-axis theorem) follows: Let C be the center of mass of the arbitrarily shaped body whose cross section is shown in the illustration. The center of mass has coordinates x_{cm} and y_{cm}. We choose the x-y plane to include C, so that z_{cm} equals zero. Consider an axis through C at right angles to the plane of the paper and another axis parallel to it through P at $(x_{cm} + a)$ and $(y_{cm} + b)$. The distance between the axes is $h = \sqrt{(a^2 + b^2)}$.



$x_i = (x_{cm} + a) - x_{cm} = a$

$y_i = (y_{cm} + b) - y_{cm} = b$

$h = \sqrt{[a^2 + b^2]}$; $h^2 = a^2 + b^2$ (Pythagoras theorem)

Then the square of the distance (h^2) of a particle from the axis through C, is $x_i^2 + y_i^2$, where x_i and y_i measure the coordinates of a mass element m_i relative to the axis through C. The square of its distance from an axis through P is $(x_i - a)^2 + (y_i - b)^2$. Hence the rotational inertia about an axis through P is

$I = \sum m_i r^2 = \sum m_i[(x_i - a)^2 + (y_i - b)^2]$

$= \sum m_i[(x_i^2 + y_i^2) - 2a \sum m_i x_i - 2b \sum m_i y_i + (a^2 + b^2) \sum m_i$

From the definition of center of mass, $\sum m_i x_{cm} = \sum m_i y_{cm} = 0$.

So that the two middle terms are zero. The first term is simply the rotational inertia about an axis though the center of mass I_{cm} and the last term is Mh^2; $[(a^2 + b^2)\sum m_i]$. Hence it follows, $I = I_{cm} + Mh^2$.

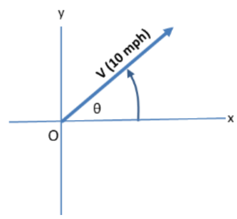
[2]**ADDENDUM 1**

Refresher on scalars, vectors and unit vectors.

A **scalar** describes a mathematic or physics quantity that is expressed by a single number and has only magnitude but not direction. For example, a particle weighing 10 grams (direction is not essential).

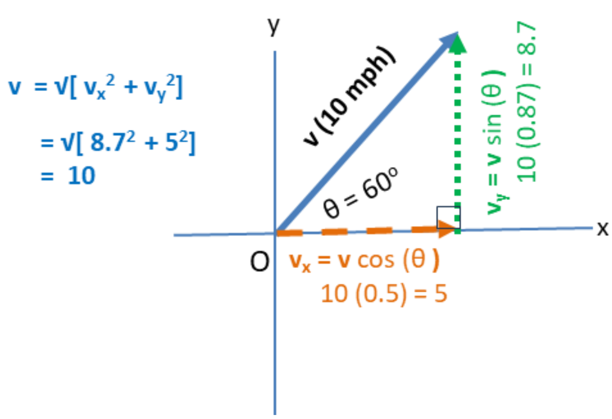
A particle weighing 10 grams

A **vector** (meaning 'carrier', moving an element from point A to point B) describes a mathematic or physics quantity that is not expressed by a single number but has both magnitude and direction, such as displacements, forces and velocity. Example, a particle traveling at 10 mph East. A vector is represented by a geometric arrow in space describing a path from Point A to Point B, the length of which is typically proportional to its magnitude and its direction expressed by citing an angle.



A vector is typically written as a bold letter such as **v** or $\vec{v}$ and may also be expressed by a tuple.

A vector is made up of x, y (if two dimensional) and z (if three dimensional) vector components. For a two dimensional vector, the components are determined by taking the sine and cosine of the angle times the vector (since the vector and its components form a right triangle) and if the x and y components are known, the vector determined by using Pythagorean theorem.



A **tuple** is a way to express the vector, **v**, and in this two-dimensional example, is written as:

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} \quad \begin{pmatrix} 5 \\ 8.7 \end{pmatrix}$$

(Move in the positive x direction the # of units
Move in the positive y direction the # of units)

Unit vector is a way to express vectors (by normalizing them, divide vector components by the vector such that the resulting vector numbers are 1 or less) in a simpler form that makes vector mathematics (such as adding vectors), much simpler. A unit vector has an absolute value or length of "1" (a single unit digit). The unit vector, for the two-dimensional example, is typically written with a 'hat' or carot as $\hat{i}$ ("i-hat") = 1, for the x component and $\hat{j}$ = 1, for the y direction. The vector, **v**, can be written with unit vectors, as

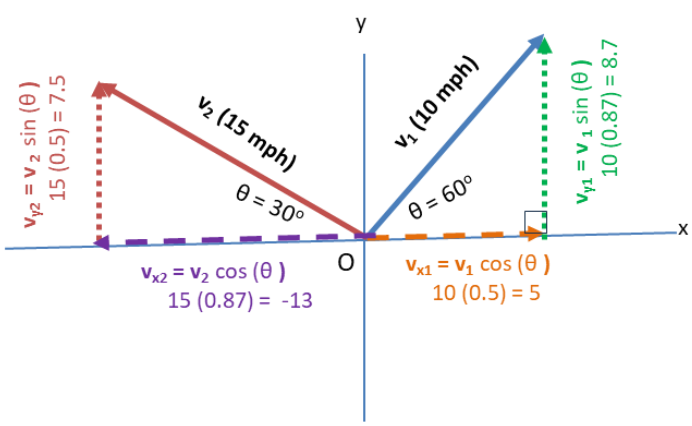
v = $\hat{i}$ (5) + $\hat{j}$ (8.7) or $\begin{pmatrix} \hat{i} & 5 \\ \hat{j} & 8.7 \end{pmatrix}$.

To convert the vector, in this example **v** = 10, to a unit vector, divide each of its x and y components by the vector, such that $\mathbf{v}_x$ = 5/10 = 0.5 and $\mathbf{v}_y$ = 8.7/10 = 0.87 and with the Pythagorean theorem, we confirm that **v** = 1 (normalized unit vector objective) =

$$\mathbf{v} = \sqrt{[\mathbf{v}_x^2 + \mathbf{v}_y^2]}$$

$$= \sqrt{[0.5^2 + 0.87^2]} = 1$$

As an example of simplified mathematical vector operation using unit vectors, assume adding the two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, illustrated in the below example:



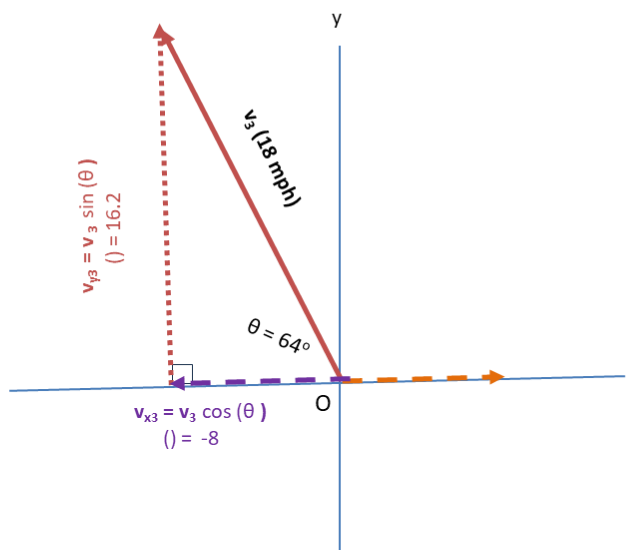
$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ = [$\hat{i}$ (5) + $\hat{j}$ (8.7)] + [$\hat{i}$ (-13) + $\hat{j}$ (7.5)] = [$\hat{i}$ (5-13) + $\hat{j}$ (8.7+7.5)] =

[$\hat{i}$ (-8) + $\hat{j}$ (16.2)]

$$\mathbf{v}_3 = \sqrt{[\mathbf{v}_{x3}^2 + \mathbf{v}_{y3}^2]}$$

$$= \sqrt{[-8^2 + 16.2^2]} = 18$$

The angle for $\mathbf{v}_3$ is determined by arccos θ = -8/18 = 64^0 or arcsin 16.2/18 = 64^0.



The resultant vector $\mathbf{v_3}$ makes sense since the x component is negative and the y component increased.